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Insurance. – (Life)
– England. – (Statistics)
– Wales. – (Statistics)

THE

DOCTRINE

OF

ANNUITIES and ASSURANCES

ON

LIVES and SURVIVORSHIPS,

STATED AND EXPLAINED.

By WILLIAM MORGAN,
ACTUARY to the Society for EQUITABLE ASSURANCES
on LIVES and SURVIVORSHIPS.

To which is added,

AN INTRODUCTION,

Addreſſed to the SOCIETY.

ALSO

An ESSAY on the PRESENT STATE of
POPULATION in ENGLAND and WALES.

By the Reverend Dr. PRICE.

LONDON:
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ASTOR, LENOX AND
TILDEN FOUNDATIONS.

INTRODUCTION.

CONTAINING

Obſervations addreſſed to the Society for Equitable Aſſurances on Lives and Survivorſhips.

HAVING encouraged Mr. *Morgan* to undertake this work, and having alſo, by his deſire, reviſed it with ſome care, I think myſelf obliged to introduce it to the notice of the public, by giving the following account of its contents, and of the views with which it has been compoſed.

The *firſt* chapter contains an explanation of the nature of Aſſurances on Lives and Survivorſhips; together with a particular account of the Society for making ſuch Aſſurances, and of the methods which have been taken to determine how far it has been hitherto a gainer by the buſineſs it has tranſacted.

The *ſecond* chapter contains an explanation of the Doctrine of Life-Annuities in general, and of the principles on which their values are calculated. At the end of this chapter, an account is given of a method of expediting all calculations of the values of Life-Annuities, which muſt, I think, be very acceptable to all who have ever employed themſelves in making ſuch calculations.

In theſe two chapters Mr. *Morgan* has ſtudied to render the ſubjects of which he treats as intelligible as poſſible to perſons who may be unacquainted with mathematics.

The *third* chapter contains a complete account of the Rules for ſolving all queſtions concerning the values (in ſingle and annual payments) of all Annuities, whether in poſſeſſion or expectation; and of all Reverſionary Intereſts depending on any *one*, *two*, or *three* lives, or on any Survivorſhips among them, either for *Terms* or for *Perpetuity*. Several of theſe queſtions are incapable of being anſwered without entering into the abſtruſeſt parts of mathematics. Moſt of them have never been before anſwered; and yet there ariſes in many caſes a neceſſity of anſwering them, in order to determine correctly the values of Reverſionary Intereſts depending on Survivorſhips; and ſome of them are often occurring in practice at the office whoſe buſineſs Mr. *Morgan* tranſacts.——All the Solutions are given in words at length, with ſuitable examples; but the mathematical inveſtigations have been thrown together into the Appendix; where every rule or direction, about which any doubt can be entertained, is particularly demonſtrated.—Every competent judge who will examine this chapter, and compare it with the demonſtrations in the Appendix, muſt admire Mr. *Morgan*'s attention and ſkill; and ſee that it contains a valuable addition to this part of ſcience.

At the cloſe of the whole, I have, with Mr. *Morgan*'s leave, inſerted an eſſay, containing an account of the preſent State of Population in this kingdom; and the intention of which is, to ſtate a great variety of facts which ſeem to prove, that while neighbouring kingdoms are increaſing, this kingdom has for many years been decreaſing faſt, in conſequence of the operation of ſome of the worſt cauſes of depopulation; and, particularly, our increaſed debts and luxury.

Such are the contents of this work; and it is now offered to the public chiefly for the following reaſons.

Firſt. In the courſe of the traffic of this kingdom, and particularly the law tranſactions of it, ſuch queſtions as thoſe contained in this work are continually occurring; and, in numberleſs caſes, it is impoſſible to give right deciſions, or to make an equitable diſtribution of property, according to the real value of different claims and intereſts, without obtaining accurate anſwers to them. The experience of many years has taught me that there are great miſtakes committed in anſwering theſe queſtions, by which many perſons are injured in their property. It ſeems, therefore, that it may be of particular uſe to the public to be informed where to apply for ſolutions that may be depended upon; and, at the ſame time, to be furniſhed with ſuch an account of the rules and principles of calculation, mathematically demonſtrated, as may remove every doubt, and be a ſufficient direction to all who may chuſe to employ themſelves in buſineſs of this kind.

But perhaps this reaſon alone might not have led Mr. *Morgan* to undertake the labour of this work. There is another reaſon by which he has been more immediately and ſtrongly influenced.—He is employed in tranſacting the buſineſs of a Society which *aſſures* all kinds of Reverſionary Annuities, and contingent intereſts, dependent on the continuance of any Lives or any Survivorſhips of any Lives beyond other Lives. In carrying on this buſineſs the Society profeſſes to regulate its demands by calculations founded on ſtrict mathematical principles. There is not any other ſociety now exiſting of the ſame kind; and its buſineſs has been, for ſome years, increaſing ſo faſt, that it is already become an object of vaſt importance to the public, and is likely ſoon to become of much more importance. It ſeems, therefore, neceſſary that the public ſhould be fully informed of its ſtate, the ſecurity it offers, the principles on which it proceeds, and the method in which it makes its calculations: And the great deſign of this publication is to give this information—an information which is rendered the more neceſſary by the multitude of *bubble*-ſocieties which were ſome time ago eſtabliſhed, and ſome of which are ſtill exiſting in this kingdom.

My acquaintance with this Society was occaſioned by the publication (in 1770) of the firſt edition of my Treatiſe on Reverſionary Payments. In that Treatiſe I ſtated particularly the plan of the Society; and at the ſame time offered ſome obſervations upon it. The Directors were pleaſed to attend to theſe obſervations; and they have ever ſince deſired my aſſiſtance, whenever they have thought I could be of uſe to them. In return for their regard and attention, as well as from a deſire to promote the public intereſt, I have been anxious about doing them all the ſervice, and giving them all the information, in my power. This led me four years ago to take into conſideration the method in which they kept their accounts, and determined annually their profits. Thinking this method too looſe, I took the liberty to propoſe another which I thought more ſtrict and deciſive. This method was approved and adopted; and by means of it the Society has been enabled to keep conſtantly under its eye the true ſtate of its affairs, the expences of management which it can afford, the progreſs of the balance in its favour from year to year, and the clear amount remaining in hand of all the profits it has made from the time of its firſt eſtabliſhment.——In this method a particular enquiry was (in 1776) made into the ſtate of the Society; and the reſult was, as explained by Mr. *Morgan* in the ſecond chapter, that it appeared, that a much ſmaller proportion of the perſons aſſured had died than the calculations ſuppoſed; and that the Society poſſeſſed *then* a ſurplus of *income* (or an income more than was neceſſary to enable it to make good its engagements and to bear the expences of management) equal to 2,400 *l.* *per annum*, nearly, and a ſurplus of *ſtock* equal to 30,000*l.* * In theſe circumſtances, the Society, not willing to continue any exorbitant profits, determined to reduce all the payments for Aſſurances on Single Lives ONE TENTH. They might have reduced them much more, but there were ſeveral reaſons which deterred the Directors from propoſing any greater reduction.

Firſt. It was proper to give the plan of the Society a longer trial before any great reductions were made in its income. It could not be depended on that the probabilities of living in the ſociety would continue ſo high as they had hitherto been; and it was to be expected, that ſeaſons of extraordinary mortality would come which would bring on the Society extraordinary expences. Its ſecurity againſt this danger lies entirely in its *ſurplus ſtock*; and it was, therefore, neceſſary, in order to give it full ſecurity, and to render it as much as poſſible a permanent benefit to the public, that this *ſurplus ſtock* ſhould be allowed to go on increaſing for ſome years longer. When I ſpeak of *full ſecurity*, I muſt be underſtood to mean all the ſecurity that property in the public funds can give. It is earneſtly to be wiſhed this was greater than it is; but though greater might be obtained, yet in an undertaking of this kind it is ſcarcely reaſonable to aim at it. The failure of the public funds will be the commencement of a new æra in this kingdom, of which, like the end of the world, we can now form no conception; and were every one to act with a view to its being ſo near as perhaps it is, there would be an end of moſt of the buſineſs and traffic of the nation.

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* The Society has *continued* to increaſe ſo faſt, that both its income and ſtock have been in the laſt three years (or from 1776 to 1779) nearly doubled. Its profits alſo have in the ſame time increaſed conſiderably, but not in the ſame proportion, on account of the reductions mentioned in this and the next pages.

Secondly. When the reduction I have mentioned was reſolved on, it was alſo agreed, that the whole overplus ſhould be returned of the payments which had been made by the members, above thoſe which they *would* have made had the reduction taken place at the time of their admiſſion. This made the Society leſs capable of bearing *any* reduction of its income; and it would have made a *greater* reduction particularly imprudent.——Different opinions have been entertained of this meaſure; but the truth is, that (however ſafe and juſt the proſperous ſtate of the Society then rendered it) it is in itſelf a meaſure of the moſt pernicious tendency, and which ought by no means to be again adopted. A repetition of it in future reductions might hurt the Society eſſentially, by withdrawing from it that ſecurity which it has been providing for many years, and bringing it back to infancy and weakneſs.——It would likewiſe be ſubverſive of its own end by making the Society incapable of proceeding to farther reductions as ſoon as it otherwiſe might; and by rendering them, when made, leſs conſiderable than they ſhould otherwiſe have been. Such a meaſure, therefore, might be no leſs detrimental to the intereſt of even the members who might be led to deſire it by the hope of preſent profit, than unkind and injurious to the Society at large.—For theſe reaſons, and in order to prevent the future bad effects which may ariſe from the late precedent, I cannot help wiſhing that a reſolution of a General Court could be procured, prohibiting every ſtep of the ſame kind hereafter.——Could ſuch a reſolution be obtained, the Directors would be at liberty to propoſe ſuch farther reductions as from time to time they might judge proper; and, particularly, they might immediately take off from all the payments for Aſſurances on Single Lives the addition of 6 *per cent.* with which theſe payments are at preſent charged in the Tables. This was once a proper charge; but, in the preſent circumſtances of the Society, it is certainly an exorbitant charge; nor does there appear to be any good reaſon for * continuing it, except the danger of being forced to another return of over-payments.

Believing as I do, that this Society is capable of being made an object of great conſequence not only to *this*, but to *other* kingdoms, and wiſhing to contribute all I can to its credit and honour, I hope I ſhall not be thought too preſuming if I proceed a little farther in theſe ſtrictures, and take the liberty to recommend to it the following obſervations and propoſals.

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* I muſt be underſtood to mention this only on the ſuppoſition that the Society will chuſe to go on uſing its preſent Tables.

Firſt. It is, particularly, to be wiſhed that the Society was furniſhed with a ſet of Tables (ſhewing the values of Aſſurances on Single Lives) more correct and elegant than thoſe it now uſes, and more accommodated to the buſineſs it carries on. Its preſent Tables have been calculated from the rate of mortality among the inhabitants of *London*, taken in the groſs, during twenty-three years, from 1728 to 1750—a period of time which included two years* of greater mortality than has been ever known in *London* ſince the plague in 1665. In conſequence of this, the values of Aſſurances on Lives are given in theſe Tables *ſomewhat* too high for the inhabitants at large of even *London* itſelf; and *much* too high for the better part of the inhabitants. There were good reaſons for beginning with ſuch Tables; but the Society being now eſtabliſhed in ſome degree of ſecurity, and having better grounds to go upon, it would, in my opinion, be right to calculate new Tables founded on obſervations which will give the values of Life-Aſſurances, not among the bulk of people in *London*, where life is particularly ſhort, but among mankind in general.—— The decrements of life at every age, as deduced by Dr. *Halley* from obſervations at *Breſlaw* in *Sileſia*, or thoſe deduced from the bills of mortality at *Northampton* or *Norwich*, are nearly the mean decrements between thoſe in great towns and in country pariſhes and villages. As the Society aſſures town and country lives indiſcriminately, theſe are the obſervations by which it ſhould be guided. But obſervations properer for its purpoſe than even theſe may be now obtained. I mean thoſe furniſhed by the *Regiſter of Mortality* eſtabliſhed a few years ago at CHESTER, under the direction of the ingenious Dr. HAYGARTH.—*Cheſter* is an old and a very healthy town, of moderate ſize, which has continued much the ſame as to populouſneſs for a long courſe of years. Theſe are circumſtances which render it a ſituation particularly fitted for ſhewing the true law that governs the waſte of human life in all its ſtages. The Regiſter which has been eſtabliſhed there is more minute and correct than any other; and it is the only one I am acquainted with that gives the difference between the chances of living among males and females, and from which it is poſſible to compute, with any degree of preciſion, the values of lives *before* five and *after* ſeventy years of age. Tables, therefore, of the values of Life-Annuities, Aſſurances and Reverſions calculated from this Regiſter, would be a valuable acquiſition; and might form a ſtandard for the buſineſs of this Society, which would be of great uſe to it. The prices of Aſſurances in ſuch Tables would indeed be found conſiderably lower than thoſe in the Tables it now uſes *. But they might be raiſed as high as ſhould appear to be neceſſary to the perfect ſafety of the Society, by making a charge upon them of 10 or 15 *per cent.* and giving them, in the Propoſals of the Society, with ſuch a charge.

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* The years 1740 and 1741.

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* Care ſhould be taken to remember that I have here in view only Aſſurances on Single Lives, the longeſt of Two or more Lives, and Joint Lives. The values of Aſſurances on Survivorſhips are, in many caſes, *greater* when the chances of life are *greater*; and it is of particular conſequence that the Society ſhould attend to this, and diſtinguiſh between the caſes in which its intereſt requires it to compute from obſervations that give the chances of life *high*, rather than from thoſe which give them *low*.

Secondly. The buſineſs of the Society requires that it ſhould be poſſeſt of Tables of the values of *two joint* lives at all ages, agreeably to the beſt obſervations, and true to at leaſt three places of decimals. Without ſuch Tables it is impoſſible to find, in many caſes, the true values of the Aſſurances to be made by the Society, and particularly of Aſſurances on *Survivorſhips for Terms*.——There are now no ſuch Tables extant. Mr. *Simpſon*'s Table in the *Select Exerciſes*, page 266, is adapted only to *London*; and gives the values only to one place of decimals. The Table in the Treatiſe on Reverſionary Payments, page 328, is calculated from Mr. *De Moivre*'s Hypotheſis; and this Hypotheſis, though agreeing nearly in the middle ſtages of life with the *Breſlaw* and *Northampton* obſervations, yet differs ſo widely from all real obſervations *before* twenty and *after* ſeventy, as to be totally improper for uſe. The Society, therefore, in order to a more perfect management of its buſineſs, ſhould direct ſuch Tables as I have deſcribed to be calculated. And were they even to go ſo far as to furniſh themſelves with Tables, equally exact, of the values of *three* *, as well as of *two Joint* Lives, their buſineſs would be ſtill more aſſiſted, and conſiderable ſervice done to this part of ſcience. The expence of ſuch calculations can be no object to them; and the moſt eaſy and expeditious method of making them has been diſtinctly deſcribed by Mr. *Morgan* in the following work.

Thirdly. I muſt take the liberty to remind the Society of the particular care neceſſary to prevent the intruſion of bad lives. It muſt be ruined, ſhould it ever become, not a reſource for the living and healthy, but a refuge for the ſick and dying. Its proſperous ſtate proves that the vigilance of the Directors has hitherto guarded it againſt this danger; but its great increaſe expoſes it to more and more of this danger; and therefore it would not perhaps be amiſs to appoint a medical aſſiſtant, whoſe particular buſineſs it ſhall be to enquire into the ſtate of health of the perſons who are offered to be aſſured. Should ſuch a regulation be the means of excluding but two or three bad lives annually, which would otherwiſe have been admitted, the expence attending it will be more than compenſated.

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* In making many calculations for the buſineſs of the Society, it is neceſſary to find the values of the *longeſt* of *two* or of *three* given lives. But this cannot be done without knowing the values of the two or of the three *joint* lives.

This, without doubt, is the *principal* danger to which the Society is expoſed. But there is another of conſiderable conſequence about which I cannot be ſilent; I mean the danger of employing unſkilful calculators. Such Tables as I have propoſed would leſſen this danger; but no Tables can entirely remove it, becauſe there are many caſes in which the values of Aſſurances cannot be obtained by Tables, without nice, and ſometimes long and laborious computations, which none but able mathematicians can make. There are indeed (as I have before intimated) no queſtions, the ſolution of which requires a ſtricter attention or more ſkill in inveſtigation than ſome in the doctrine of Aſſurances. Difficult queſtions of this kind are ſometimes brought to the Society; and the Directors, not being themſelves mathematicians, are under a neceſſity, in making their demands, of being governed by their Actuary: And ſhould he happen to be unqualified, he muſt make miſtakes, and either the Public or the Society will be injured, In ſhort. The Society can ſcarcely be ſenſible enough of the importance of both ability and probity in the ſervants it employs; nor, therefore, of the particular reaſon there is for guarding their places againſt the applications of candidates who, on any future vacancies, may endeavour to intrude themſelves by their connexions or influence.

I intended to have proceeded to ſome farther obſervations and propoſals; but I have perhaps already gone too far. Thoſe which have been offered are ſincerely meant to promote the proſperity of the Society; and they will therefore, I doubt not, be received with candour.

RICHARD PRICE.

NEWINGTON-GREEN,
May 20, 1779.

ERRATA.

Page 3, line 11, for "*thoſe of*" read "*in*"

Page 155, line 6th of the Note, for "17th Problem," read "19th Problem."

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CHARACTERS *uſed in this Work.*

$+$ Signifies that the quantity to which it is prefixed is to be added. Thus $\frac{35}{36} + \frac{25}{26}$ denotes the *ſum* ariſing from the addition of $\frac{25}{26}$ to $\frac{35}{36}$.

$-$ Serves to ſhew that the quantity which it precedes is to be ſubtracted. Thus $\frac{35}{36} - \frac{25}{26}$ denotes the *difference* between thoſe two fractions, or the *exceſs* of $\frac{35}{36}$ above $\frac{25}{26}$.

$\times$ Signifies that the quantities between which it is placed are to be multiplied together. Thus, $\frac{35}{36} \times \frac{25}{26}$ expreſſes the product ariſing from the multiplication of $\frac{35}{36}$ into $\frac{25}{26}$.

$=$ Denotes equality. Thus, $3 + 2 = 5$ ſignifies that 2 added to 3 is equal to 5.

When a line is drawn over any number of figures, it ſignifies that all thoſe figures are to be conſidered as one compound quantity. Thus, $\overline{1 - \frac{35}{36}} \times \overline{1 - \frac{25}{26}}$ denotes the product ariſing from the multiplication of $\frac{35}{36}$ ſubtracted from unity, into $\frac{25}{26}$ ſubtracted from unity.

CHAPTER I.

An Introduction to the Doctrine of Aſſurances on Lives.

SECT. I.

THE eaſe with which the value of an Aſſurance is computed depends on the number of lives involved in the calculation. In order, therefore, to give a clear idea of this ſubject, and the principles from which the ſeveral queſtions are inveſtigated, it will be neceſſary to begin with thoſe caſes where only one life is concerned.—The ſimpleſt of this kind is the Aſſurance of a ſingle life for the term of one year.—Nothing further is required here than to determine the probability of the life's failing in this time; and in proportion as this probability is greater or leſs, the value of the

Aſſurance will alſo be greater or leſs; ſo that was it *certain* that the life would fail in the year, the premium of Aſſurance would be equal to the ſum aſſured: was it an *even* chance whether it failed or not, it would be worth *half* as much: was there only *one chance out of twenty*, the premium would only be a twentieth, and ſo on.

Suppoſe, for example, that 50 perſons aged 39 agreed to aſſure 100 *l.* each on their lives for one year.—By Dr. *Halley*'s Table of Obſervations it appears that one of them will die in the year. The value of the Aſſurance, therefore, will be $\frac{1}{50}$ of 100 *l.* or 2 *l.* from each perſon; for this contribution will juſt furniſh money enough to pay the claim. But ſince the money is generally advanced at the beginning, and the claim is not paid till the end of the year, this ſum muſt be diſcounted for a year, that the ſeveral contributions, together with their intereſt, may be juſt equal to the ſum aſſured. —Had theſe perſons been 59 years of age, 2 of them would have died in the year, and the contributions would have been twice as much; had they been 70 years of age, 4 of them would have died, and conſequently the contributions would have been 4 times as much. This ſhews the reaſon why the value of an Aſſurance for a year is continually increaſing with age, till at laſt it becomes equal to the ſum aſſured.

Let the time now be 2, 3, or any certain number of years, the ſum, as before, 100 *l.* the whole price of Aſſurance to be paid down immediately, and the Claim whenever the death happens.——The principles of inveſtigation in this caſe are a little more complicated than thoſe of the former, and they will perhaps be beſt explained in the following manner:—Let 86 be ſuppoſed the extremity of life, and the difference between this period and the age of the perſon be called the *complement* of life; let the probability of living alſo decreaſe according to *De Moivre*'s Hypotheſis * in arithmetic progreſſion. Theſe things being premiſed, ſuppoſe the age of the perſon to be aſſured is 66, the complement of his life will be 20; and as the probability of his living is leſſened every year by unity, the chances for his living the firſt year will be 19 out of 20; for his living 2 years, 18 out of 20; for his living 3 years, 17 out of 20, and ſo on; from whence it follows, that the chance for his *dying* the firſt year will be 1 out of 20; for his dying the ſecond year, after ſurviving the firſt, 1 out of 19; of his dying the third year, after ſurviving the two firſt years, 1 out of 18, and ſo on. The value, therefore, of the Aſſurance the firſt year will be a *twentieth* of the ſum diſcounted for a year. Should the perſon ſurvive this period, for which he has 19 chances out of 20 (or a probability expreſſed by the fraction $\frac{19}{20}$) the value, at the beginning of the

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* In this Hypotheſis the limit or utmoſt extent of life is confined to the age of 86, and the probabilities of exiſtence are ſuppoſed to decreaſe in *arithmetic* progreſſion; that is, ſuppoſing any number living at a given age, an equal number of them will die every year, till at the age of 86 all the lives will become extinct. Thus for inſtance; if there are 56 perſons living at the age of 30, one of them, by this hypotheſis, will die annually during the term of 56 years, at which time the laſt ſurviving life will have failed:—Or in other words, 55 will be living at the end of the firſt year, 54 at the end of the ſecond year, 53 at the end of the third year, and ſo

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ſo on for 56 years; from whence it follows, that the probability of one perſon's living to the end of the firſt, or ſecond, or third year will be $\frac{55}{56}$, $\frac{54}{56}$, $\frac{53}{56}$ reſpectively; where it appears that the fractions, and conſequently the probabilities they expreſs, decreaſe *arithmetically*. Therefore if n be the difference between any given age and 86, the probability of exiſting at this age, for one, two, three, &c. years, will be properly denoted by the correſponding term of the ſeries, $\frac{n-1}{n}$, $\frac{n-2}{n}$, $\frac{n-3}{n}$, &c.

firſt year, of the Aſſurance in the ſecond year, will depend upon the contingency of his dying in that year after having ſurvived the firſt. The probability that he dies in the ſecond year, is expreſſed by the fraction $\frac{1}{19}$, which being multiplied by $\frac{19}{20}$ (the probability that he ſurvives the firſt) we have $\frac{1}{20}$ for the ratio in which the ſum diſcounted for two years is to be diminiſhed in order to obtain the value of the Aſſurance for the ſecond year. In the ſame manner, the value at the beginning of the firſt year of the Aſſurance in the third year depends upon the perſon's ſurviving the two firſt years, for which he has 18 chances out of 20, and his dying in the third year, the chance for which is 1 out of 18, theſe two fractions, or $\frac{18}{20}$ and $\frac{1}{18}$ being multiplied into each other, we ſtill ſhall have $\frac{1}{20}$ to expreſs the ratio in which the ſum, diſcounted in this caſe for three years, is to be diminiſhed. By reaſoning in the ſame way, the value of the Aſſurance during the fourth, fifth, &c. years will be conſtantly found $\frac{1}{20}$th of the ſum diſcounted for four, five, &c. years; and the amount of all theſe ſums will be the whole value of the Aſſurance in one preſent payment.—Was this Aſſurance extended to the whole duration of life, the value would be determined in a ſimilar manner, only continuing the calculations for a number of years equal to the *complement* of the life; at which time it becomes a certainty by the hypotheſis * that it will fail, and of courſe that the claim will become due.—The ſame reaſoning may be applied to every age, and any table of obſervations.—Thus; ſuppoſe the perſon is 30, and the probabilities of life as given by Dr. *Halley*. The chances for his living one year will be 523 out of 531; for his living two years, 515 out of 531; for his living three years, 507 out of 531, and ſo on. The chances for his *dying* the firſt year are 8 out of 531; for his dying the ſecond year, after ſurviving the firſt, 8 out of 523; for his dying the third year, after ſurviving the two firſt years, 8 out of 507, &c. The value of the Aſſurance, therefore, the firſt year will be 100 *l.* diſcounted for a year, and diminiſhed in the ratio of $\frac{8}{531}$; the value of it the ſecond year, at the beginning of the firſt, will be 100 *l.* diſcounted for two years, and diminiſhed in the ratio of $\frac{523}{531}\times\frac{8}{523}$ or $\frac{8}{531}$; the third year it will be 100 *l.* diſcounted for three years, and diminiſhed in the ratio of $\frac{507}{531}$ multiplied into $\frac{8}{507}$, or $\frac{8}{531}$; which will continue invariably the ſame as long as the probabilities of life decreaſe in arithmetic progreſſion; but this not being always the caſe, the ratio will conſequently be different; hence in the preſent inſtance the probability of the perſon's living 14 years is expreſſed by the fraction $\frac{407}{531}$ *, and the probability of his dying the 15th, after ſurviving the firſt fourteen years, by $\frac{10}{407}$; theſe two fractions, therefore, being multiplied into each other, we have $\frac{10}{531}$ for the ratio in which the ſum (properly diſcounted) is to be diminiſhed the 15th year. In other years this ratio will be found as much leſs as it is here greater than $\frac{8}{531}$; ſo that upon the whole it coincides very nearly with Mr. *De Moivre*'s hypotheſis. In what manner the amount of the ſeveral ſums (diſcounted and diminiſhed for the reſpective years) is determined, may be ſeen in Dr. *Price*'s Treatiſe, Queſtion 10th and 14th †; and in *Simſon*'s Select Exerciſes, Problem 26. My deſign being only to give a general idea of the *principles* from which the values of Aſſurances are inveſtigated, it is unneceſſary to be more particular.

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* See Note, page 3.

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* See Note under Prob. 4th, chap. 3d.

† Theſe two Queſtions are the ſame with the eighth and ninth Problems in the third chapter of this work.

If, inſtead of, a *ſingle* payment, the value of an Aſſurance of 100 *l.* for any term of years was required in *annual* payments, the anſwer may be obtained by reaſoning in the following manner :—Suppoſe 50 perſons aged 36 had agreed to aſſure 100 *l.* each on their lives from year to year for any certain term, that is, by equal contributions to diſcharge whatever claims may happen in the courſe of every year. By *De Moivre's* hypotheſis one of them will die in the firſt year and become a claimant at the end of the year. The value of the Aſſurance from each perſon, therefore, will be 2 *l.*; but if the money has been advanced at the beginning of the year this value will be leſs, or only 2 *l.* diſcounted for a year.—The ſecond year there will be 49 perſons alive, and one in 49 will die by the ſame hypotheſis, conſequently one will become again a claimant of 100 *l.* in the courſe of the ſecond year; but a contribution of 2 *l.* each from 49 perſons will not be ſufficient to diſcharge ſuch a claim; at the beginning of the ſecond year, therefore, they ought to contribute a little more than they did at the beginning of the firſt year;—and for the ſame reaſon, thoſe of them that will live to the *third* year ought to contribute more than they did the *ſecond* year in order to make good the ſame Aſſurance, and ſo on for every year during the whole term. If the price of Aſſurance is paid, not in annual contributions continually *increaſing* from year to year, but in one *fixed* annual contribution, it is obvious that this contribution ought to be higher than the contribution due for an Aſſurance in the firſt year of the term, and lower than the ſame contribution in the laſt year. What this *mean* ſhould be, may be determined by the Rule in Dr. *Price's* 14th Queſtion.

Had there been 50 perſons at the age of 61, two of them would have died by the ſame hypotheſis every year, and conſequently the contributions muſt have been *double* what they were at the age of 36. From whence the reaſon may be inferred, why the difference between the price of Aſſurance for one year, and the ſhort term of ſix or ſeven years, is not ſo conſiderable on a *young* life as it is on an *old* one; for the probabilities of dying being leſs in one caſe than the other, the prices of Aſſurance muſt be leſs, and conſequently the *differences* leſs.

Were theſe annual payments to be continued during the *whole* of life, that is, was it an Aſſurance of 100 *l.* to be paid whenever the life fails, the reaſoning will be exactly ſimilar to that in the preceding caſe. But the *mean* of all the payments will be much more conſiderable here; for thoſe contributions muſt increaſe annually, till at laſt they become equal to the ſum aſſured. This follows from the nature of the hypotheſis; which ſuppoſing that one perſon will die every year till they are all extinct, reduces them at the extremity of life to one ſurvivor, whoſe contribution muſt conſequently be the ſum aſſured.—It ſhould, however, be obſerved, that the mean between the firſt and laſt payment is not to be underſtood as a ſimple *arithmetical* mean, or the *half* of the ſum of thoſe payments, but ſuch a conſtant annual ſum, which muſt ſo far exceed the value of an Aſſurance for the firſt year as to produce a *ſurplus* that, in conſequence of being laid up, ſhall form a capital, which, together with its intereſt added to the annual payments, ſhall be ſufficient to diſcharge the claims as they become payable; ſo that when the laſt life drops, juſt ſo much of the capital ſhall be left as, together with the intereſt due, ſhall be equal to one claim. Such a mean as here deſcribed may be found in Dr. *Price*'s Treatiſe on Reverſionary Payments, Queſtion 10th *.

Theſe are the chief principles from which the values of all Aſſurances on *ſingle* lives are deduced. Thoſe of the ſame kind on the *joint* continuance of the lives, or the *longeſt* of any number of lives, are found in a ſimilar manner. I would only obſerve that as, in ſuppoſing any two events, the chance is univerſally *leſs* for the happening of *both* of them than of *one*, ſo the value of an Aſſurance on two joint lives which depends upon ſuch a contingency muſt be *greater* than the value of an Aſſurance on the ſingle life of one of them; for in the one caſe the claim is paid if they do not *both* live to the end of the term, that is, if *two* events do not happen; in the other caſe, it is paid if one does not live to the end of the term, or if *one* event does not happen. For the ſame reaſon the value of an Aſſurance for the longeſt of two, or any number of lives, will be *leſs* than on a ſingle life; for here the claim is not paid unleſs two or more events *do* happen, that is, unleſs two or more lives fail in the term. So that it is plain the greater the number of *joint* lives that are aſſured, the *greater* will be the value of the Aſſurance; and on the contrary, the greater the number that are aſſured during the *longeſt liver*, the *leſs* will be the value of the Aſſurance. The ſolutions of all caſes of this kind being inveſtigated from the ſame principles with thoſe of ſingle lives, they are conſequently anſwered by the ſame rules in Dr. *Price*'s Treatiſe, Queſtions 10th and 14th *.

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* This is the ſame with the eighth Prob. in the third chapter of this work.

Aſſurances on *Survivorſhips* are much more complicated, eſpecially where ſeveral lives are concerned; but at preſent I ſhall only explain thoſe of two lives, as they are by far the moſt common; and being properly underſtood, it will not be difficult to reaſon from the ſame principles in caſes where a greater number of lives is concerned.—— To begin with the moſt ſimple, let us ſuppoſe the Aſſurance made only for one year on the contingency of one given life's ſurviving another. Let the age of the perſon to be ſurvived be 50, and of the other 30;

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* See Cor. to Prob. 8th, and Cor. to Prob. 9th, in the 3d chapter of this work.

that is, a given ſum is to be paid in caſe the *former* ſhould die *before* the latter in one year.—By the hypotheſis*, the complement of a perſon aged 50 is 36, and that of a perſon aged 30 is 56. It follows, therefore, that the youngeſt perſon has 55 chances out of 56 for *living*, and the oldeſt perſon one chance out of 36 for *dying*, in the year. The payment of the claim depends upon either of two events; firſt, that they ſhall *both* die in the year and the youngeſt ſhall die laſt; ſecondly, that the oldeſt perſon *only* ſhall die in the year. The probability that they ſhall both † die is expreſſed by the two fractions $\frac{1}{36}$ and $\frac{1}{56}$ multiplied into each other, or $\frac{1}{36 \times 56}$. But that one in particular ſhall die *laſt* is an equal chance. *Half* of this fraction therefore, or $\frac{1}{2 \times 36 \times 56}$, will be the value of the expectation on the firſt of the two events. The ſecond, which is by far the moſt conſiderable, will be expreſſed by the probability of the oldeſt perſon's ſon's *dying*, or $\frac{1}{36}$, multiplied into the probability of the youngeſt perſon's *living*, or $\frac{55}{56}$. The ſum of theſe two fractions therefore, or $\frac{1}{2\times36\times56}$ added to $\frac{55}{56\times56}$, $=\frac{111}{4032}$, will expreſs the *whole* ratio in which the ſum propoſed to be aſſured, after being diſcounted for a year, is to be diminiſhed. By reaſoning in the ſame manner, the value of any other Aſſurance of this kind may be determined.

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* See note, page 3.

† It is here taken for granted that the reader is ſo far acquainted with the doctrine of Chances as to know, that the probability that any number of independent events will all happen is the product of all their probabilities ſeparately taken multiplied by one another.

If the term of the Aſſurance is extended to two, three, &c. years, or to the extremity of life, the principles by which the ſolution is inveſtigated are exactly ſimilar, provided the value is required in *one preſent payment*. Thus, the chance that the oldeſt ſhall die in any year is conſtantly $\frac{1}{36}$, and that the youngeſt ſhall die $\frac{1}{56}$; this appears by reaſoning as in page 5. Half the fraction $\frac{1}{36\times56}$ will therefore always expreſs the value of the expectation on the contingency of their both *dying* in the ſame year, and one in particular dying *laſt*. The other expectation after the firſt year depends upon the youngeſt perſon's *living* two, three, &c. years, and the oldeſt perſon's *dying* in the firſt, ſecond, third, &c. years, and is determined by conſtantly multiplying $\frac{1}{36}$, expreſſing the latter probability, into the ſeveral fractions expreſſing the former probability. For the youngeſt perſon's living two years, there are, by the Hypotheſis, 54 chances out of 56; for his living three years, 53 chances out of 56, and ſo on. The fractions $\frac{54}{56}$, $\frac{53}{56}$, &c. will therefore expreſs the probability of his living two, three, &c. years. Theſe being ſeverally multiplied by $\frac{1}{36}$ and added to $\frac{1}{2\times36\times56}$, expreſſing the expectation on the firſt contingency, we ſhall have $\frac{109}{4032}$ for the ratio in which the ſum to be aſſured, after being diſcounted for two years, is to be diminiſhed in the ſecond year; $\frac{107}{4032}$ for the ratio in which, after being diſcounted for three years, it is to be diminiſhed in the third year, and ſo on. The ſum, therefore, of all theſe terms is the whole value of the Aſſurance in one preſent payment. If it is only for a limited time, the number of terms muſt be equal to the number of years; if it is for life, the terms muſt be continued to the extremity of the oldeſt life. How theſe are ſummed up and reduced into general rules may be ſeen in *Simſon*'s Select Exerciſes, Problem 32d—and Dr. *Price*'s Treatiſe, Queſtion 11th *.

In computing the values of Aſſurances on Survivorſhips in *annual* payments from the ſame principles with thoſe on ſingle lives, the reaſoning is much more complicated. Suppoſe a given ſum to be paid at the death of a perſon aged 61 ſhould that happen before the death of another aged 36, and that 50 perſons, all aged 61, had agreed to aſſure 100 *l.* each on their lives dependent on this contingency. By *De Moivre*'s Hypotheſis one will die out of 50 at the age of 36 in the firſt year, and two out of the ſame number at the age of 61.—*One* claim, therefore, will certainly become due in this year, towards which a contribution of 2 *l.* from each perſon muſt be paid.—That the other does not become due depends upon two events; firſt, that the youngeſt life which drops was one of thoſe aſſured againſt either of the two oldeſt lives which failed in this year; ſecondly, that provided this was the caſe, the youngeſt perſon alſo died *before* the oldeſt one in the year. That the firſt event happens, there are two chances out of 50, or a probability which may be expreſſed by the fraction $\frac{2}{50}$, or $\frac{1}{25}$; that the ſecond event happens is an *even* chance, which is expreſſed by $\frac{1}{2}$. Theſe two fractions being multiplied into each other, we ſhall have $\frac{1}{2\times25}=\frac{1}{50}$ expreſſing the whole probability that the ſecond claim does not become due; that is, $\frac{1}{50}$th, or 2 *l.* muſt be deducted from this 100 *l.* and the remaining 98 *l.* divided into equal contributions, which being added to the former contribution of 2 *l.* will make the whole of the firſt year's Aſſurance 3 *l.* 19*s.* 2*d*; and as this is ſuppoſed payable at the beginning, and the claim not till the end of the year, it muſt be diſcounted for one year.——In the ſecond year, there will be very nearly three Aſſurances leſs than in the firſt year; and as the deaths among the old and young lives continually bear the ſame proportion to each other, there will again be nearly two claims to be paid in this year; which being divided among fewer members, will render the contributions from each perſon ſo much the higher. In the third year there will be nearly ſix Aſſurances leſs than in the firſt, and three leſs than in the ſecond year. The claims will, however, be much the ſame, and conſequently the contributions higher than in the preceding year. By reaſoning in the ſame manner, it will be found that the annual payments are continually increaſing till they become greateſt in the laſt year; from whence it is obvious, that the true value of a *fixed* annual payment is determined by taking a mean, which will ſo far exceed the firſt yearly payment as to produce a *ſurplus*, that, with its accumulating intereſt, will be ſufficient to diſcharge the claims of the laſt years. I have not attempted to aſcertain with *exactneſs* how much the contributions increaſe and the claims diminiſh in every year, for the reaſoning becomes ſo perplexed and complicated as to render it unfit for the preſent purpoſe. It appears, however, from this general view, that the contributions increaſe much faſter than the claims diminiſh; from whence a proof ariſes that the *laſt* payment is the moſt conſiderable, and that a *fixed* annual premium ſhould be ſuch a mean as I have deſcribed. The method of finding this mean in all caſes may be ſeen in Dr. *Price*'s Treatiſe, Queſtion 11th.

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* See Problem 16th, chap. 3d, of this work.

The annual payments, both for the Aſſurances on Survivorſhips and Single Lives, are generally determined by dividing the value of the Aſſurance in one payment by the value of an annuity on the joint or ſingle lives; and the reaſon of this may be eaſily inferred.—Thus; let the ſingle premium of Aſſurance on the above Survivorſhip be 50 *l.* If, inſtead of one payment, it is agreed to take a fixed *annual* payment during the continuance of the joint lives, it is plain the value of this will depend on the number of years they will exiſt; that is, on the number of years which, one with another, all joint lives at thoſe ages do exiſt together, ſome exiſting as much *longer*, as others exiſt a *ſhorter* time; or, in other words, on the *mean* duration of the joint continuance of the lives, called by *De Moivre*, and other writers on the ſubject, the *expectation* of the joint lives. In the preſent inſtance, this expectation is juſt equal to ten years and a half; ſo that the annual payment will be 4 *l.* 15 *s.* 3 *d.*; for ten payments and a half, of 4 *l.* 15 *s.* 3 *d.* each, will be equal to 50 *l*, ſuppoſing the perſon aſſured to make no intereſt of his money; but as the contrary is always underſtood, it follows that 50 *l.* received in this manner, cannot be an equivalent to the ſame ſum received in one preſent payment; for the latter, in 10½ years, at 4 per cent. compound intereſt, will accumulate to

75 *l.*

75 *l.* 9 *s.* 6 *d.* whereas the former, under the ſame circumſtances, will not exceed 60*l.* 13*s.* 4*d.*—The preceding annual payments, therefore, ſhould be increaſed; and the true value of an annuity on the joint lives being nearly 7½ (allowing a diſcount at 4 per *cent.*) this, and not 10½, will be the ſum by which the *ſingle* premium of 50 *l.* muſt be divided, in order to find the *annual* premium equivalent to it*. A ſimilar reaſoning may be applied to Survivorſhips *for terms*, and to Aſſurances on Single Lives *for terms* and *continuance*.

In examining the principles upon which the calculations of Aſſurances are founded, I have generally choſen to make uſe of *De Moivre*'s Hypotheſis, as being the moſt ſimple. The ſame reaſoning, however, may be formed from any table of obſervations; many of which correſpond very nearly with the hypotheſis.

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* This calculation ſuppoſes the firſt payment to be made at the *end* of the year; but if it is made at the *beginning* of the year (which is commonly the caſe) unity muſt be added to the value of the joint lives. This, in the preſent inſtance, would reduce the annual premium to 5*l.* 17*s.* 8*d.* which, without the addition of unity, is 6*l.* 13*s.* 3*d.*

SECT.

SECT. II.

On the different Methods of determining the State of a Society, whoſe Buſineſs conſiſts in making Aſſurances on Lives.

HAVING conſidered the principles from which the values of Aſſurances in general are inveſtigated, let us now ſuppoſe a ſociety eſtabliſhed for making ſuch Aſſurances. It is obvious, that ſo far as it does not regulate itſelf according to theſe principles, it muſt proceed in the dark, and be either requiring exorbitant contributions on the one hand, or laying the foundation for future diſappointment on the other. It becomes, therefore, of the laſt conſequence that a ſociety of this kind ſhould always be able to determine its real ſtate, and I will beg leave to give the following methods for this purpoſe.

I. Let an exact account be taken of the proportion of aſſured perſons who have died annually, or of the decrements of life at the different ages of the members, from the eſtabliſhment of the ſociety to the preſent time. If theſe decrements of life ſhould appear to be *leſs* than thoſe in the tables from which the calculations are made, a demonſtration will ariſe of the thriving ſtate of the ſociety, and it will enjoy advantages equivalent to the exceſs of the decrements of life in the tables above thoſe in the ſociety.—If, on the contrary, theſe decrements of life ſhould appear to be greater than thoſe in the tables, or that a larger number dies than the calculations ſuppoſe, the ſociety will ſuſtain loſſes proportionable to this difference.——The probabilities of life in a ſociety ſhould be always a little *higher* than thoſe in the tables, in order to balance the neceſſary expences of management, and to enable it to lay up againſt any particular ſeaſon of mortality; from whence it follows that, ſhould the probabilities of life in the ſociety be *lower* than thoſe in the tables, its permanency muſt be ſoon deſtroyed, if others are not uſed which make the decrements of life more conſiderable.

II. The ſhorteſt method of trying the ſtate of the ſociety (as far as it aſſures *ſingle* lives) is by comparing the amount of the *annual payments* which have been made for ſuch aſſurances, with the amount of the annual *claims* derived from them; and the following general rule may be given for this purpoſe:

"Find the ſum of all the annual payments for aſſurances on the *whole duration* of ſingle lives. Find alſo the ſum of all the claims grounded upon theſe payments. In like manner, find the ſum of all the annual payments for aſſurances on ſingle lives for any term not exceeding ſeven years. Find alſo the ſum of all the claims derived from theſe laſt payments. If in the *latter* caſe, the ſum of all the payments is equal to the ſum of all the claims, and in the *former* caſe, *greater* in the proportion of three to two, it will appear that the payments are ſufficient to ſupport the engagements of the ſociety, ſuppoſing no expences of management; and as much as the payments are in a higher proportion than this to the claims, ſo much profit will the ſociety enjoy."

The principles from which this rule is derived will appear from the following obſervations.——Suppoſe a body of people all 40 years of age to form themſelves into a ſociety for the purpoſe of aſſuring 100*l.* each on their lives for their whole duration. Suppoſe alſo that they keep up the ſociety to the ſame conſtant number by admitting continually new members at the age of 40 in the room of ſuch old members as die off.

It is plain in theſe circumſtances that the whole collective body of members will be continually growing older till their mean age comes to be as great as poſſible. To this greateſt mean age they will attain in a number of years equal to the difference between the age at admiſſion, or 40, and the extremity of life, that is in 46 years, ſuppoſing the extremity of life to be 86. During this period the number of members dying annually will be continually increaſing, till at the end of it about twice as many will die annually as died at the firſt eſtabliſhment of the ſociety. The claims, conſequently, will alſo increaſe annually till they are doubled. Had, therefore, the payments neceſſary to ſupport the Aſſurances been made from year to year in the manner deſcribed in page 8, they muſt have been always increaſing till they became double; but if the value of the Aſſurance is paid by contributing an annual ſum conſtantly the ſame, this annual ſum muſt be a mean between the payment due the firſt year and the double payment which would become neceſſary at the end of 46 years, agreeable to what has been obſerved in page 19.— According to *De Moivre*'s Hypotheſis * the value of an Aſſurance of 100 *l.* for the whole duration of a life aged 40 is in annual payments (the firſt to be made immediately) 3 *l.* 4 *s.* reckoning intereſt at 4 per cent. The value by the ſame Hypotheſis of the ſame Aſſurance for a ſingle year is 2 *l.* 1 *s.* 10 *d.* At firſt therefore in the ſociety there will be an annual *ſurplus* equal to 1 *l.* 2 *s.* 2 *d.* from each member over and above the growing intereſt of his payments. This ſurplus will decreaſe continually as the ſociety grows older, and in 46 years it will vaniſh; but it will be ſufficient in that time to raiſe a capital, the annual intereſt of which added to 3 *l.* 4 *s.* from each member will make up a ſum equal to that to which the annual claims will increaſe in 46 years. Let the number of members be 1000, all when admitted 40 years of age, and having 100 *l.* each aſſured to them for the whole duration of their lives at a *fixed* annual payment. By *De Moivre*'s Hypotheſis the number of deaths the firſt year will be 21.74. Each of thoſe deaths will produce a claim of 100 *l.* and therefore the ſociety will at the end of the year have the ſum of 2174 *l.* to pay; but the contributions of 3 *l.* 4 *s.* each will amount to 3200 *l.* and 128 *l.* being a year's intereſt for 3200 *l.* the total, or 3328 *l.* will be the whole income of the ſociety the firſt year. The difference between this ſum and 2174*l.* (or 1154*l.*) will be the firſt year's ſurplus to be laid up and improved at compound intereſt.—A ſurplus a little *leſs* will be left the ſecond year, becauſe more will die that year. Another ſurplus ſtill leſs will be left the third year for the ſame reaſon, and ſo on for 46 years. The annual claims having been all this time in a ſtate of increaſe, they will at laſt come to be 43½. The annual ſum then payable by the ſociety will be 4350 *l.* or 1150 *l.* *more* than the annual contributions, but at that period the capital formed from the annual ſurpluſſes will have increaſed to ſuch a ſum as at 4 per cent. will yield this intereſt; that is, it will have increaſed to the ſum of 28,750 *l.*

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* See note, page 3.

Now it ſhould be obſerved in the preceding account, that 3200 *l.* the amount of the contributions of 1000 members all 40 at admiſſion, is to the amount of the claims almoſt exactly as three to two. In ſome ages this proportion may be found to be lower than three to two; in others it may be found higher; the rule therefore which has been given may be conſidered as ſufficiently exact when applied to the circumſtances of a ſociety which aſſures lives at all ages. If this ſociety has been of any ſtanding, it affords a teſt rather too ſevere; for we have ſeen that the proportion of the annual payments to the annual claims muſt be continually decreaſing till in a courſe of years the claims become *equal* to the payments, and afterwards *greater*. But during the firſt eight or ten years from the commencement of the ſociety this rule will be pretty near the truth, and as it errs after this time on the ſafe ſide, no bad conſequences can ariſe from having recourſe to it. ——From what has been ſaid, it is, however, eaſy to determine with exactneſs in any period what proportion the claims ſhould bear to the annual payments; for in 23 years they will become equal, and in 46 years the claims will exeed the payments in the proportion of 4350 *l.* to 3200 *l.* See page 26.—As to Aſſurances for terms not exceeding ſeven years, I have obſerved that the claims derived from them ought in all periods of the ſociety to be nearly equal to the payments. In order to prove this, let us take as before a life at the age of 40. 100 *l.* aſſured on ſuch a life for a year is worth, I have ſaid, in preſent payment, 2 *l.* 1 *s.* 10 *d.* From a thouſand ſuch Aſſurances will ariſe at the end of the year 21.739 claims of 100 *l.* The ſociety therefore at the end of the year will have a debt to pay of 2173 *l.* 18 *s.* but the payments at the beginning of the year at 2 *l.* 1 *s.* 10 *d.* each amounted only to 2090 *l.* 6 *s.* In the caſe therefore of Aſſurances for a ſingle year the claims ought a little to exceed the payments; and the reaſon is, that the payments being made at the beginning of the year, they are ſuppoſed in the calculation to bear a year's intereſt, which is to be added to them in order to make up a ſum equal to the claims at the end of the year.—Again 100 *l.* aſſured on the ſame life for two years is worth 2 *l.* 2 *s.* 3 *d.* in two equal payments, the firſt to be made immediately and the ſecond a year hence. 1000 ſuch payments the firſt year will make 2113 *l.* but the claims, it has juſt appeared, will amount to 2173 *l.* 18 *s.* and the ſecond year they will be a little more; but (the number of ſuch Aſſurances being ſuppoſed to be kept up) the payments will be the ſame. In the caſe therefore of Aſſurances on ſingle lives for two years the payments will both years fall ſhort of the claims. An Aſſurance on the ſame life for three years is worth 2 *l.*

2 *l.* 2 *s.* 9 *d.* and 1000 ſuch payments the firſt year will make 2136 *l.* but the claims in this year will amount to 2173 *l.* 18 *s.* the ſecond year to a little more, and the third to ſtill more.—An Aſſurance on the ſame life for four years is worth 2 *l.* 3 *s.* 2 *d.* 1000 ſuch payments the firſt year will make 2158*l.* which ſtill fall ſhort of the claims in this year, and of courſe muſt fall ſhort of them in the three ſucceeding years. In an Aſſurance for five years the annual payments will be 2 *l.* 3 *s.* 7 *d.* 1000 ſuch payments will make 2181 *l.* which ſum exceeds the claims in the firſt year but will be leſs in the four following years. In Aſſurances for ſix years the payments will exceed the claims the firſt year, be nearly equal to them the ſecond year, and leſs the three ſucceeding years. The like is true in Aſſurances for ſeven years; and what has been now applied to the age of 40 may, without any great difference, be applied to any other age. Take therefore all Aſſurances together made at different times in the ſociety, ſome for one, ſome for two, three, &c. years, it may be expected that the claims and payments will be nearly equal, and if the latter exceed the former, the ſociety muſt be ſo far a gainer.

III. The third and laſt method is the moſt deciſive, and determines not only whether the ſociety is in a thriving ſtate, but likewiſe of what ſtock it is poſſeſſed after paying to all the different members the value of their intereſts in their reſpective Aſſurances. It may be inferred from the ſecond method (ſuppóſing the Aſſurances paid for at the beginning of the year) that the ſtock which ſhould be raiſed from Aſſurances on ſingle lives for *terms* from one to ſeven years is nearly one year's payment for thoſe Aſſurances; and it has appeared alſo that from the payments of a ſtanding body of 1000 members aſſuring 100 *l.* each and all aged 40 at admiſſion, the capital which ſhould be raiſed in 46 years is 28,750 *l.* ſuppoſing intereſt at 4 per cent. and the probabilities of life as they are according to *De Moivre*'s Hypotheſis. By applying theſe rules, therefore, under proper reſtrictions, the ſurplus ſtock of the ſociety may be nearly aſcertained ſo far as relates to Aſſurances on *ſingle lives*; but this method does not extend to Aſſurances on the *joint* lives, the *longeſt* of two lives, or on *ſurvivorſhips*. Theſe are indeed in common with all the other Aſſurances included in the firſt method, but as it can never be proved from hence what the ſociety is really poſſeſſed of after fulfilling its engagements, and as it will be proper that an inſtitution of this conſequence be examined by every poſſible teſt to determine its ſtability, I ſhall beg leave to lay down the following rule for this purpoſe.—" Find the *preſent* value in *ſingle* " payments of all the Aſſurances ſubſiſting " in the ſociety.—Find likewiſe ſo much " of all the different annual payments as is " proportionable to that part of the year " which is yet to come, before the next " payments become due. Add both theſe " values together, and let their ſum be re- " ſerved.—Find next the *preſent* values in " *ſingle* payments of all the *annual* pre- " miums due upon the ſeveral Aſſurances; " to theſe add the preſent ſtock or capital " of the ſociety; and as far as this laſt " ſum *exceeds* the ſum reſerved above, of ſo " much *ſurplus* ſtock will the ſociety be " poſſeſſed; but if it falls ſhort of this re- " ſerved ſum, a demonſtration will ariſe " that the plan of the ſociety is inade- " quate."

EXAMPLE.

EXAMPLE.

Suppoſe there are 1000 Aſſurances for 100 *l.* each on the whole duration of life, and that the annual premiums are 3 *l* 4 *s.* each, or in the whole 3200 *l.*—Let theſe members be now 50 years of age; and the value of an Aſſurance in one preſent payment on ſuch a life being by *De Moivre's* Hypotheſis and at 4 per cent. 52 *l.* 10 *s.* 5½ *d.* 1000 ſuch Aſſurances will be worth 52,523 *l.* Suppoſe half a year has elapſed ſince the laſt annual payments were made, half theſe therefore, or 1600 *l.* muſt be added to the preceding ſum, which will increaſe it to 54,123 *l.*——The value of an Annuity on a life of 50 at 4 per cent. is worth 11⅓ year's purchaſe nearly, and as the annual payment is to be made in half a year, this value muſt be increaſed by half-unity, which will make it equal to 11⅚.—If 3200 *l.* be multiplied by 11⅚ we ſhall have 37866 *l.* for the preſent value of the annual premiums in one payment.——In the ſame manner let there be 500 Aſſurances of 100 *l.* each for a term of years, of which two are unexpired, at 1095 *l.* per ann.—Let the age of theſe lives be 43 years, and the preſent value in one payment of thoſe Aſſurances on their lives for two years is 2199 *l.*—Suppoſe their annual payments juſt due, nothing in this caſe is therefore to be added.——The value of a life of 43 for one year is worth $\frac{939}{1000}$ year's purchaſe. If 1095 *l.* be multiplied by this fraction, and the annual payment juſt due be added to the product, we ſhall have 2123 *l.* for the whole value of the premiums in one preſent payment.——Again. Let there be 20 Aſſurances of 100 *l.* each on the joint continuance of two lives aged 45 and 55 at the annual premiums collectively of 109 *l.* 16 *s.*—The value of thoſe Aſſurances in one *preſent* payment will be 1304 *l.* 6 *s.*[*] Suppoſe the premiums to have been paid three quarters of a year ago; one quarter of them, or 27 *l.* 9 *s*, muſt be added to the preceding value which will make the whole 1331 *l.* 15 *s.* By multiplying the annual premiums by the value of the joint lives, increaſed in this caſe by three fourths of unity, as the next payment is to be made in a quarter of a year, we ſhall have 965 *l.* 12 *s.* for the preſent value of thoſe premiums in one payment.——In the ſame manner may the Aſſurances on the longeſt of two lives be found. Let there be ten ſuch Aſſurances for 100 *l.* each on two lives whoſe reſpective ages are 50 and 60, at the annual premiums collectively of 18 *l.* 10 *s.* Suppoſing their payments juſt made, the whole year's premium is here to be added to the preſent values of thoſe Aſſurances, which will be found equal to 469 *l.* 12 *s.*—The premiums multiplied by $13\frac{1}{4}$ (or the value of an annuity on the longeſt of two lives aged 50 and 60) will produce 245 *l.* 2 *s.* for the value of thoſe premiums in one preſent payment.——Laſtly, let there be 60 Aſſurances of 100 *l.* each on the ſurvivorſhip of one life beyond another. Let the preſent age of the perſons to ſurvive be 50, and of the perſons to be ſurvived 60, and let the annual premiums collectively be 171 *l.*—Suppoſe one quarter of a year to have elapſed ſince the laſt premiums were paid. 128 *l.* 5 *s.* is therefore to be added to 2578 *l.* 16 *s.* which by the rules in chap. 3d may be found to be the preſent values of thoſe Aſſurances in one payment, and the total will be 2707 *l.* 1 *s.* The premiums multiplied by $7\frac{3}{10}$, or the value of two joint lives whoſe reſpective ages are 50 and 60 increaſed by one fourth of unity, will produce 1248*l.* 6*s.* for the value of thoſe premiums in one preſent payment. Now the values of all theſe Aſſurances and proportional parts being added together will amount to 60,830*l.* 8*s.*—The values of all the annual payments due for theſe Aſſurances being alſo added together will amount to 42,448*l.* If 18,382*l.* 8*s*, the difference between thoſe two ſums, be not found in the ſtock of the ſociety, or, which is the ſame thing, if the ſtock of the ſociety added to 42,448*l.* be not equal to 60,830*l.* 8*s*, a proof will enſue that the ſociety is not able to fulfil its engagements. As much therefore as the two former ſums exceed the latter, of ſo much *ſurplus* ſtock will the ſociety be poſſeſſed.

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[*] The method of finding theſe values, and in general the values of all Aſſurances on Lives and Survivorſhips for terms and perpetuity, will be particularly explained in the third chapter.

Having been ſo particular in this example, and the principles of thoſe calculations being ſo obvious, any further explanation of them can hardly be neceſſary *. I am ſenſible that a work of this kind would be attended with great labour where the ſociety is numerous; for the values of all the different Aſſurances that do not agree in time, date, and in the ages of the perſons aſſured, muſt be calculated *ſeparately*; but no labour ſhould be avoided that is proper to throw light on the affairs of ſuch a ſociety, and to give a more clear and ſatisfactory view of its ſtate.

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* Suppoſe the ſociety had engaged ten years ago to pay 100*l.* at the death of a perſon aged 40. This perſon being now 50, the value of ſuch a ſum in *preſent* money is 52*l.* 10*s.* 5*d.* which would therefore expreſs the value of his preſent intereſt in the reverſion, ſuppoſing him to have completed his payment for it. But if he has agreed to pay for it by an annual premium of 3*l.* 4*s.* during life, it will follow that he *owes* as much as ſuch an annual premium dependent on his life is worth.—A life

I ſhall conclude this ſection with obſerving that the ſtate of the *Equitable Society* (whoſe buſineſs conſiſts in making all the different aſſurances I have deſcribed, and who

life of 50 is worth 11⅓ year's purchaſe, or 12⅓ as the firſt payment is to be made immediately, the preſent value, therefore, of what this perſon is indebted will be 3*l.* 4*s.* multiplied by 12⅓ or 39*l.* 7*s.* Conſequently his intereſt, or the balance in his favour will be no more than the difference between 52*l.* 10*s.* 5*d.* and 39*l.* 7*s.* or 13*l.* 3*s.* 5*d.* But if he applies for this balance at any other time than when his annual payment becomes due; it is plain, ſince the values of lives in the table ſuppoſe the firſt payment not to be made till the end of a year, that the annual premium muſt be multiplied by 11⅓ increaſed by as much leſs than unity as what remains of the year is leſs than the whole year. By proceeding in the ſame manner with all Aſſurances for Terms, Survivorſhips, &c. in the ſociety and adding thoſe differences together, we have the whole of what it is indebted to its members; and ſo far as this falls ſhort of, or exceeds its *ſtock*, ſo far will the ſociety be in a precarious, or in a flouriſhing condition.

calculate their premiums from the principles explained in ſection I.) has lately been examined by each of the methods here propoſed, and I ſhall now lay the reſult of thoſe examinations before the public.

By the firſt method it has been found that during the term of nine years (from January 1768 to January 1777) the proportion of deaths in the Society to thoſe in the Table of Obſervations for London, from which its premiums have been calculated, are,

From the age of 20 to 30 as 7 to 17
30 to 40 as 3 to 5
40 to 50 as 1 to 2
50 to 60 as 2 to 3
60 to 70 as 14 to 11

Or in all ages from 20 to 70 as two to three nearly; that is, where the tables have ſuppoſed three claims would have happened, only two have taken place in the ſociety.—By comparing the decrements of life in this ſociety with thoſe in Dr. *Halley*'s table, which agree very nearly with *De Moivre*'s Hypotheſis, they are found to be,

From the age of 20 to 30 as 5 to 9
30 to 40 as 5 to 6
40 to 50 as 6 to 7
50 to 60 as 5 to 6
60 to 70 as 5 to 3

Or in all ages from 20 to 70 as 11 to 13 nearly.

By the *ſecond* method, or by determining the proportion of the claims to the premiums for Aſſurances on Single Lives, it appears that the claims in this ſociety have on an average for theſe laſt nine years been 3115 *l. per ann. leſs* than they ſhould have been; that is, the Society has enjoyed a clear *annual* profit during this term of 3115*l.* from that part of its buſineſs only which conſiſts in aſſuring Single Lives.

By the *third* method, which is the moſt certain and deciſive, the ſurplus ſtock of this Society is proved by calculating even by the *London* Table, to be conſiderably above 30,000 *l.*—In conſequence of thoſe concurring teſtimonies of the flouriſhing ſtate of this Society it was reſolved in June 1777 to reduce the premiums of Aſſurance one *tenth*, and to return to every member the exceſs of the payments he has actually made above the payments he would have made, had the reduction taken place from the time of his admiſſion.—Thoſe reductions and returns ought undoubtedly to be conſidered as a diminution both of the annual profits and ſurplus ſtock of the Society; and it may alſo be obſerved that if the calculations had been made *after* the premiums were reduced, their preſent values would have been *leſs*, which by increaſing the difference between them and the preſent values of the Aſſurances in *one* payment, would have likewiſe proportionably diminiſhed the ſurplus. But theſe objections have very little weight.—The annual profits of the Society conſiderably exceed the reduction, together with all the expences of management; nor are the returns greater than the exceſs of the ſurplus ſtock above 30,000 *l.* Had alſo the preſent values of the *ſingle* and *annual* payments been calculated by Dr. *Halley*'s Table (as it is evident they ſhould have been*,) the former would have *decreaſed*, and the latter *increaſed* in a much larger proportion than one tenth.—The accumulating intereſt, therefore, on a ſurplus ſtock of 30,000 *l.* together with the annual profits which ſtill remain over and above the expences of management, perfectly ſecure this Society from any future danger, and render its late reſolutions not only *ſafe*, but highly *juſt* and *neceſſary*.

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[*] See page 37.

CHAP. II.

An Introduction to the Doctrine of Annuities.

SECT. I.

Of Annuities on Lives.

THE value of a *Life Annuity*, like that of an *Aſſurance*, is more or leſs eaſily determined in proportion to the number of lives, on the continuance of which the annuity depends. Where, therefore, only *one* life is concerned the operation will be moſt ſimple, and if the Annuity is limited to a ſingle year it becomes the exact converſe of that for an *Aſſurance* during the ſame time; ſince in the former, a given ſum is to be paid if the life *exiſts* one year, in the latter the ſum is to be paid if the life *fails* in the year; ſo that if it were an *even* chance whether it would exiſt through a year or not, the value of an Aſſurance and Annuity would be equal: and in proportion as the chance of *living* exceeds that of *dying* in the year, in the ſame proportion will the value of the *Annuity* exceed the value of the *Aſſurance.*—Suppoſe, for example, that A. and his heirs were entitled to 50 Annuities of 100 *l.* each on the ſingle lives of as many perſons, whoſe common age was 39; that is, that they were to receive 100 *l.* from every one of 50 perſons of this age at the end of every year while they lived. By Dr. *Halley*'s Table of Obſervations it appears that 49 out of the 50 will live one year; the value, therefore, of the firſt payments of thoſe annuities is 100 *l.* multiplied into 49, (or 5000 *l.* the ſum of all the payments, leſſened in the ratio of 49 to 50) which is equal to 4900 *l.* But as this money is not payable till the *end* of the year, it muſt be diſcounted for a year, ſo that the preſent value together with the intereſt may in this time juſt accumulate to 4900 *l.*——From hence it follows that the value of the firſt payment of an Annuity in the preſent inſtance exceeds that of an Aſſurance for a year in the ratio of 49 to 1; for in page 2 it is proved that the premium for aſſuring thoſe ſeveral ſums on 50 ſuch lives is only 100 *l.* diſcounted for a year.—Had the Annuity been upon 50 ſingle lives whoſe common age was 59, two perſons would have died by Dr. *Halley*'s table in the year, ſo that the firſt payment would have been 100 *l.* multiplied into 48 (or 5000 *l.* leſſened in the ratio of 48 to 50) which is equal to 4800 *l.* and in page 2 it is proved that the premium for aſſuring thoſe ſums is 200 *l.* *; hence the firſt payment of an *Annuity* on a life of 59 exceeds an *Aſſurance* for a year on the ſame life only in the ratio of 48 to 2 or 24 to 1.—At the age of 89 the number of thoſe that die *exceeds* the number of thoſe that live one year; in which caſe the value of the firſt payment of an Annuity becomes *leſs* than the value of an Aſſurance for a year of a ſum equal to the Annuity. It may be obſerved from thoſe calculations, that the ſum of the values of the firſt payment of an Annuity and Aſſurance for one year is always equal to the value of a ſum equal to the Annuity to be received *certainly* at the end of the year. This indeed is ſelf-evident, for if a given ſum is to be received whether a life *does* or *does not* exiſt to the end of a year, one or other of theſe two events muſt happen, and conſequently the expectation on *both* of them is the expectation of a certainty.

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* It ſhould be obſerved that both theſe ſums of 4800 *l.* and 290 *l.* are to be diſcounted for a year.

The value of the different payments of a life annuity for two, three, or any number of years is determined much in the ſame manner.—Suppoſe the age of the perſon to be 66, and the probabilities of life to decreaſe according to *De Moivre*'s hypotheſis*, the chances in this caſe for the annuitant's living to the end of the firſt year are 19 out of 20, and the value, therefore, of 1 *l.* to be received at the end of a year on the contingency of his living ſo long will be expreſſed by the ſum diſcounted for a year and multiplied into the fraction $\frac{19}{20}$.——In the ſecond year the chances for his living to the end of that year are only 18 out of 20, ſo that the value of 1 *l.* to be received at the end of this year will be expreſſed by the ſum diſcounted for two years and multiplied into the fraction $\frac{18}{20}$.—The value of 1 *l.* to be received at the end of the third year, by reaſoning in the ſame manner, will be expreſſed by the ſum diſcounted for three years and multiplied into the fraction $\frac{17}{20}$.——By proceeding thus through any number of

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* See this hypotheſis explained in note, page 3.

terms and adding the ſeveral products together, we ſhall have the value of an Annuity of 1 *l.* upon this life for a number of *years* equal to the number of terms; and if we continue the operations for 20 terms, it is evident that the numerator of the 20th term vaniſhes, or, which is the ſame thing, that the life in 20 years beomes extinct; and conſequently that the ſum of all the preceding 19 terms will expreſs the value of an annuity upon the *whole life* of a perſon whoſe age is 66.—This reaſoning may be extended to all ages, as well as to any table of obſervations.—Let the age, for inſtance, be 20, and the probabilities of life as in Dr. *Halley*'s table. It appears from thence that out of 598 perſons at the beginning of the firſt year, no more than 592 will live to the beginning of the ſecond year, 586 to the beginning of the third year, 579 to the beginning of the fourth year, and ſo on. The value, therefore, of the firſt payment of an annuity on a life whoſe age is 20 is expreſſed by the ſum diſcounted for a year and multiplied into the fraction $\frac{592}{598}$. The value of the payment for the ſecond year is expreſſed by the ſum diſcounted for two years and multiplied into the fraction $\frac{586}{598}$.

The value of the payment for the third year is expreſſed by the ſum diſcounted for three years and multiplied into the fraction $\frac{579}{598}$. —If the operations are continued to 71 terms it is evident from the table that the numerator of the fraction which conſtitutes the 71ſt term vaniſhes; and conſequently the ſum of the different products for 70 terms will expreſs the value of the Annuity for the whole duration of a life aged 20.

This ſubject may be otherwiſe explained in the following manner.—Suppoſe a Life-Annuity of 1 *l.* payable yearly to every one of 515 perſons all now aged 20, the firſt payment of which is to be made a year hence. It appears from the *Northampton* Table of Obſervations * that only 507 of theſe perſons will be living at the end of a year, and conſequently that the money to be then paid will be only 507 *l.* The preſent value, therefore, of the firſt payment of the Annuities will be the ſum which now put out to intereſt will increaſe in a year to 507 *l.* that is, it is 507 *l.* diſcounted for a year, or 487 *l.* 10 *s.*; for this ſum added to its intereſt for a year (reckoned at 4 per *cent.*) will juſt make up 507 *l.*—From the ſame table it appears further that of 515 perſons living at 20 years of age only 499 will be living at the end of two years. The preſent value, therefore, of the ſecond payment of the Annuities will be the ſum which being now put out to compound intereſt at 4 *per cent.* will increaſe in *two* years to 499 *l.*—This ſum is 461 *l.* 7 *s.* In like manner 491, 483, 475, &c. being the number living at the end of 3, 4, 5, &c. years, the value of the 3d, 4th, 5th, &c. payments of the Annuities will be 491 *l.* 483 *l.* 475 *l.* &c. diſcounted for 3, 4, 5, &c. years reſpectively, and continued to the year in which all the lives become extinct.——The total of all theſe values is 8204 *l.* which therefore is the ſum that would be ſufficient, if improved at 4 *per cent.* to make good the payment of an Annuity of 1 *l.* for life to every one of 515 perſons aged 20, according to the *Northampton* Table of Obſervations. The value, therefore, of ſuch an Annuity payable to only one ſuch perſon muſt be the 515th part of 8204*l.* or 15*l.* 18*s.* 6*d.*—This ſeems to be all extremely clear; but he that would ſee it drawn out at length and fully explained ſhould conſult a book entitled *Calculations deduced from firſt Principles*, &c.*

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* See table 3d, Appendix.

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* The author of this work (Mr. *Dale*) has lately publiſhed a SUPPLEMENT to it; and he deſerves the higheſt

Suppoſing the probabilities of life to decreaſe in arithmetic progreſſion agreeable to *De Moivre*'s hypotheſis, the ſum of the ſeveral products is eaſily calculated by Queſt. 56, vol. 2, *Dodſon*'s Mathematical Repoſitory, or by Prob. 1ſt, Cor. 5th, *Simpſon*'s Doctrine of Annuities, or by Prob. 2d in the next chapter.—But thoſe rules cannot be applied to *tables of obſervation* with ſufficient accuracy, nor is it poſſible that any general rules, *ſtrictly true*, can be delivered for this purpoſe. Under theſe circumſtances, therefore, it becomes neceſſary to

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higheſt praiſe for the pains he has taken to detect and expoſe thoſe bubbles, which were ſome time ago eſtabliſhed under the name of *Societies for the Benefit of Old Age*. Moſt of theſe ſocieties, convinced of their miſtake, have diſſolved themſelves. But there is one ſtill left, on which none of the calls of juſtice and humanity have been able to make any proper impreſſion. I mean the *Laudable Society of Annuitants for the Benefit of old Age*, whoſe office is held at the bottom of *Bartholomew Lane*. In oppoſition to the plaineſt evidence, this ſociety goes on to offer *double* the Annuity it can afford to pay; and the late tranſactions in it (as related by Mr. *Dale* in his Supplement juſt mentioned) exhibit an inſtance of ſuch an obſtinate and wilful perſeverance in impoſition, as hath ſeldom been equalled.—I am ſorry to add, that this cenſure is applicable to another Annuity ſociety called alſo LAUDABLE, but in reality PERNICIOUS, as many ſuffering widows will ſome time or other experience.

proceed through every fraction to the extremity of life, which in younger ages is attended with vaſt labour.—Theſe difficulties, however, may in a great meaſure be obviated; and I ſhall in the ſubſequent ſection lay down a method by which the values of Annuities upon lives of all ages may be computed nearly in the ſame length of time with a *ſingle* Annuity upon a life of the youngeſt age.

The values of Annuities upon the *joint* continuance of two lives are determined from the ſame principles with thoſe on ſingle lives.—Suppoſe the ages of two perſons were 50 and 60. Their complements by the hypotheſis (that is, the difference between their ages and 86 *) will be 36 and 26, and the chances for their living ſingly the firſt year will be 35 out of 36, and 25 out of 26; conſequently the probabilities of their *both* living to the end of this year will be expreſſed by the two fractions $\frac{35}{36}$ and $\frac{25}{26}$ multiplied into each other †. In the ſame manner the probabilities of their *both* living to the end of the ſecond year will be expreſſed by the two fractions $\frac{34}{36}$ and $\frac{24}{26}$ multiplied into each other. The probabilities of their *both* living to the end of the third year by the two fractions $\frac{33}{36}$ and $\frac{23}{26}$ multiplied into each other, and ſo on for 26 years; at the end of which the numerator of the fraction expreſſing the probability of the oldeſt life's exiſtence vaniſhes. Thoſe ſeveral fractions, therefore, multiplied into the ſum properly diſcounted, and then added together, will give the whole value of the Annuity upon the *joint* continuance of two ſuch lives.

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* See note, page 3.

† See note, page 13.

The values of Annuities on the *joint* continuance of three or any other number of lives are found in the ſame manner; the ſeveral fractions expreſſing their reſpective probabilities each year being multiplied into one another, and alſo into the ſum diſcounted as above, and the aggregate of the products ariſing from thoſe multiplications continued to the extremity of the *oldeſt* life will give the whole value of the Annuity.

It is plain from what has been obſerved that the value of an Annuity upon a *ſingle* life muſt be always greater than that of an Annuity upon *two joint* lives of the ſame age; for in the firſt, the money is to be received if only *one* event happens, in the ſecond it is not to be received unleſs that event and another both happen.——For the ſame reaſon an Annuity upon *two joint* lives is worth more than an Annuity upon *three joint* lives of the ſame age, inaſmuch as the money in the one caſe is to be received if only *two* events happen, in the other it is not to be received unleſs thoſe two events and one more all happen. Univerſally therefore, the *greater* the number of lives the *leſs* will be the value of the Annuity on their *joint* continuance; and this number may be increaſed ſo far as even to render ſuch an Annuity of *no* value.—In the caſe of an equal decrement of life, that is, ſuppoſing that out of any number living at any given age an equal number will die every year till all are dead, the values of thoſe Annuities for two or three joint lives may be calculated by Queſt. 64 and 69, vol. 2. of *Dodſon*'s Mathematical Repoſitory, and by Prob. 1, Cor. 5, of *Simpſon*'s Doctrine of Annuities, or for two joint lives by Prob. 3. in the next chapter.—When the calculations are to be made from a table of obſervations theſe rules are of no uſe, and the *real* values can only be determined by a tedious multiplication of all the terms.—This requiring too much labour, mathematicians have generally contented themſelves with approximations to the true values.—Mr. *Simpſon* in his Select Exerciſes, Prob. 9th *, has given a rule for determining the value of three joint lives from the given value of two joint lives with tolerable preciſion. But it is evident that the more accurate the *latter* value is given, the nearer an approximation will Mr. *Simpſon*'s rule be to the real value of the *former*.—On this and many other accounts, tables of the *exact* values of two joint lives are exceedingly deſirable; and ſhould any perſon undertake ſuch an uſeful and neceſſary work, he may perform it with a vaſt abridgment of labour by the method laid down in the following ſection.

With reſpect to Annuities on the *longeſt* of any number of lives, the reaſoning is not ſo ſimple; and if there are more than two lives concerned, it becomes ſo complicated as not to be capable of being explained in a few words. I ſhall, therefore, confine myſelf to the ſingle caſe of two lives, which being properly underſtood, the ſame principles may be applied without much difficulty to any greater number of lives.

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* This is the ſame with the 5th Problem of the following chapter.

Suppoſe the ages of the two perſons are 50 and 60, and, for the ſake of more perſpicuity, that the decrements of life are according to *De Moivre*'s hypotheſis *.— The chance of the youngeſt perſon's living one year is expreſſed by $\frac{35}{36}$, and the ſame chance with regard to the oldeſt by $\frac{25}{26}$. The chance of the youngeſt perſon's *not* living one year is $\frac{35}{36}$ ſubtracted from unity †, or $\overline{1-\frac{35}{36}}$, and the ſame chance with regard to the oldeſt is alſo $\overline{1-\frac{25}{26}}$. Theſe two expreſſions being multiplied into each other will give the probability of their *both* dying in the year, which is equal to $1-\frac{35}{36}-\frac{25}{26}+\frac{35\times25}{26\times36}$. For ſuppoſe the firſt fraction $1-\frac{35}{36}$ to be multiplied into unity, then it will ſtill remain equal to $1-\frac{35}{36}$. Let it again be multiplied into $\frac{25}{26}$ and it will become equal to $\frac{25}{26}-\frac{25\times35}{26\times36}$. But $\frac{25}{26}$ was a *negative* quantity, therefore the ſigns of the two laſt fractions muſt be changed, or wrote down $\frac{25}{26}+\frac{25\times35}{26\times36}$. Theſe being added to $1-\frac{35}{36}$, will give $1-\frac{35}{36}-\frac{25}{26}+\frac{25\times35}{26\times36}$ for the whole product of $\overline{1-\frac{35}{36}}$ into $\overline{1-\frac{25}{26}}$.——Now ſince thoſe expreſſions give the probability that the two lives ſhall *both* fail in the year, it follows that if they are ſubtracted from unity*, they will give the probability that they ſhall *not* both fail in the year; that is, that one of them at leaſt will live to the end of this year. In order to ſubtract $1-\frac{35}{36}-\frac{25}{26}+\frac{25\times35}{26\times36}$ from unity it is evident that nothing more is neceſſary than to change the ſigns of this expreſſion and then add it to unity; in which caſe we ſhall have $1-1+\frac{35}{36}+\frac{25}{26}-\frac{25\times35}{36\times26}=\frac{35}{36}+\frac{25}{26}-\frac{25\times35}{26\times36}$ for the true value of the above probability.—In the ſame manner the chance of one of them at leaſt living to the end of the ſecond year is expreſſed by $\frac{34}{36}+\frac{24}{26}-\frac{24\times34}{26\times36}$.—The chance of one of them at leaſt living to the end of the third year by $\frac{33}{36}+\frac{23}{26}-\frac{23\times33}{26\times36}$, and ſo on for a number of years equal to the complement of the youngeſt life (or the difference between its preſent age and 86,) as it is plain that *all* the fractions will not vaniſh ſooner.——If thoſe ſeveral expreſſions are reſpectively multiplied into the ſum diſcounted for 1, 2, 3, &c. years, and the products are added together, we ſhall have the whole value of the Annuity for the continuance of the longeſt of two lives whoſe ages are 50 and 60.—By reaſoning in the ſame manner the value of an Annuity on the longeſt of any two lives, and by any table of obſervations, may be determined. But if the values of the ſingle and joint lives are given, the value of an Annuity during the continuance of the *longeſt* of them is immediately found; for it appears from what has been ſaid on ſingle and joint lives, pag. 43 and 48, that $\frac{35}{36}$ multiplied into 1 *l.* diſcounted for a year expreſſes the value of the firſt payment of an Annuity on a life whoſe age is 50, that $\frac{25}{26}$ multiplied into the ſame ſum expreſſes the like value for a life of 60, and that $\frac{25\times35}{26\times36}$ multiplied as above expreſſes the value of the firſt payment of an Annuity on the joint continuance of thoſe lives. It follows, therefore, that $\overline{\frac{35}{36}+\frac{25}{26}-\frac{25\times35}{26\times36}}$ multiplied into 1 *l.* diſcounted for a year, and expreſſing the value of an Annuity on the *longeſt* of two lives for one year, is equal to the difference between the value of the *joint* lives and the *ſum* of the values of the *ſingle* lives for the ſame term.—The like may be obſerved with reſpect to the ſecond, third, &c. years; from whence we have the following general rule:—“ Subtract the value “ of an Annuity on the *joint* lives from the “ ſum of the values of an Annuity on the “ *ſingle* lives, and the *remainder* will give “ the value of an Annuity on the continu- “ ance of the *longeſt* of two ſuch lives *.”

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* See note, page 3.

† A certainty is always denoted by *unity*; and, therefore, the *difference* between the probability of an event's *happening* and this certainty, muſt expreſs the probability of the ſame event's *not* happening.

It ſhould be farther remembered here, that the probability that any two independent events will both happen is always the probability that the one will happen multiplied by the probability that the other will happen. For more particular information on this ſubject the reader is referred to *Simpſon*'s and *De Moivre*'s Doctrine o Chances, and alſo to the 23d and following Queſtion in the 2d vol. of *Dodſon*'s Mathematical Repoſitory.

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* See note, page 52.

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* EXAMPLE. Suppoſe the value of an Annuity is required on the longeſt of two lives whoſe ages are 40 and 50.—By Table 5th in the Appendix, the value of a ſingle life of 40 is 13.196, and by the ſame Table the value of a ſingle life of 50 is 11. 344.—Their *ſum* therefore is 24.54, from which 8.911 (the value of the *joint* lives

If the lives are increaſed to 3, 4, 5, or any other number, the value of an Annuity on the *longeſt* of them is ſtill found from the values of the *ſingle* and *joint* lives, but in a different manner.—Thoſe caſes, however, as I have already ſaid, being ſo complicated as not to admit of being explained in a few words, I ſhall beg leave to refer the reader for further information on the ſubject to the ſecond Problem and its Corrollaries in *Simpſon*'s Doctrine of Annuities.

SECT. II.

On the Method of calculating Tables of the Values of Annuities on Single and Joint Lives.

WAS it *certain* that a perſon of a given age would live to the end of a year, the value of an Annuity of 1 *l.* on ſuch a life would be the preſent ſum that would increaſe in a year to the value of a life one year older, together with the value of the ſingle payment of 1 *l.* to be made at the end

lives of 40 and 50 by Table 6th in the Appendix) being ſubtracted, we have 15.629 for the number of years purchaſe required.

end of a year; that is, it would be 1 *l.* together with the value of a liſe aged one year older than the given life, multiplied by the value of 1 *l.* payable at the end of a year.——Call the value of a life one year older than the given life *N*, and the value of 1 *l.* payable at the end of a year $\frac{1}{r}$; then will the value of an Annuity on the given life on the ſuppoſition of a *certainty* be $\frac{1}{r}+\frac{1}{r}\times N=\frac{1}{r}\times\overline{1+N}$.——But the fact is, that it is *uncertain* whether the given life will exiſt to the end of the year or not, this laſt value, therefore, muſt be diminiſhed in the proportion of this uncertainty; that is, it muſt be multiplied by the probability that the given life will ſurvive one year, or ſuppoſing $\frac{b}{a}$ to expreſs this probability, it will be $\frac{b}{ar}\times\overline{1+N}$.

This Theorem has been otherwiſe demonſtrated by Mr. *Simpſon* in his book on Life-Annuities, Cor. 7th, Prob. 1ſt, and by Dr. *Price* in his Treatiſe on Reverſionary Payments, Note O of the Appendix *.—Its great utility will appear from the following examples.—Suppoſe the probabilities of life as they are given by Dr. *Price* from the Bills of Mortality at Northampton *, and the rate of intereſt 4 *per cent.* or $r = 1.04$. By reaſoning as in page 43, the value of a life aged 91 will be expreſſed by the ſingle fraction $\frac{1}{2} \times .9615384 = .4807692$ †. The value of a life one year younger by this theorem will be $\frac{2}{4 \times 1.04} \times 1.4807692 = .7119$. The value of a life two years younger by the ſame Theorem will be $\frac{4}{6 \times 1.04} \times 1.7119 = 1.09737$. The value of a life three years younger will be $\frac{6}{8 \times 1.04} \times 2.09737 = 1.5125$, and if we proceed in this manner, the value of every younger life may be deduced from that next preceding; nor will the number of multiplications neceſſary to determine the values of all the lives much exceed the number of thoſe which muſt otherwiſe be uſed for the ſingle value of the youngeſt life.

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* See alſo an *algebraical* demonſtration of this Theorem in note A, of the Appendix to this work.

In computing a table of the values of Annuities according to this or any other theorem it cannot but be particularly agreeable to have the truth of our operations continually proved as we go on.——I do not know that any rule has been delivered for this purpoſe, on which account I ſhall beg leave to offer the following.—" Find the " value of 1 *l.* payable if the youngeſt life " ſhould live to the age of the oldeſt in the " table. Find next the value of 1 *l.* pay- " able if the youngeſt life ſhould live to an " age one year leſs than the oldeſt in the " table. Multiply this laſt value by the " value of an Annuity of 1 *l.* on the oldeſt " life but one in the table, found in the " manner juſt directed, and if the product " is equal to the value of 1 *l.* payable on " the firſt contingency mentioned in this " rule, a demonſtration ariſes that the va- " lue of the life is right. Again, find the " value of 1 *l.* payable if the youngeſt life " ſhould live to an age two years leſs than " the oldeſt life in the table. Multiply " this by the value of an Annuity of 1 *l.* " on a life likewiſe two years younger than " the oldeſt life, found from the value juſt " calculated of a life one year younger, and " if the product is equal to the ſum of the " two values of 1 *l.* payable if the youngeſt " life ſhould continue to the age of the " oldeſt, and to the age of the life one " year leſs than the oldeſt, the value of this " ſecond life will be alſo right. In the " ſame manner let the three values of 1 *l.*

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* See Table 3d, Appendix.

† See Table 1ſt, Appendix.

“ payable if the youngeſt life ſhould continue to the ſeveral ages of the oldeſt, the ſecond, and third lives, be added together, and if they are equal to the product ariſing from the multiplication of the value of an Annuity on the life three years younger than the oldeſt (found from the value laſt calculated of a life two years younger) into the value of 1 *l.* payable if the youngeſt life ſhould continue to an age alſo three years leſs than the oldeſt, a demonſtration will be given that the value of this third life has been rightly found.—Univerſally; if the ſeveral values of 1 *l.* payable if the youngeſt life in the table ſhould live to the age of the oldeſt, ſecond, third, fourth, &c. lives be added together, and are found equal reſpectively to the products ariſing from the value of an Annuity on a life one, two, three, four, &c. years younger than the oldeſt life multiplied into the value of 1 *l.* payable if the youngeſt life ſhould continue to an age one, two, three, four, &c. years leſs than the oldeſt life; a proof will ariſe of the exactneſs of the operations as far as they have proceeded; and when we have gone in this manne through all the ages in the table, and th

" ſum of the ſeveral values of 1 *l.* payable
" if the youngeſt life ſhould continue to the
" ages of all the older lives is found *equal*
" to the value, found laſt of all, of the
" youngeſt life, we have a concluſive proof
" that the values of lives at all ages in the
" table have been computed without com-
" mitting any miſtake, and are exact at leaſt
" as far, or to as many places of decimals,
" as the equality reaches."

This rule may perhaps be better underſtood from the following examples.——Let the probabilities of life be as in the Northampton Table *, and the rate of intereſt 4 per cent.—By reaſoning as in page 44 †, the value of 1 *l.* payable if a life aged one year lives to the age of 92 is expreſſed by the fraction $\frac{1}{849}$ multiplied into 1 *l.* diſcounted for 91 years ‡, or .028182, which is equal to .0000332. Again, the value of 1 *l.* payable if the ſame life continues to the age of 91 is $\frac{2}{849}$ multiplied into 1 *l.* diſcounted for 90 years, or .029309, which is equal to .000069. The value of a Life

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* See Table 3d in the Appendix.

† See alſo the note under Problem 4th, chap. 3d.

‡ See the value of 1 *l.* diſcounted for any number of years in Table 1ſt, Appendix.

Annuity of 1 *l.* on a perſon aged 91 has been found in page 58 to be equal to .4807692, and this being multiplied into .000069 gives .0000332, which has likewiſe been juſt found to be the value of 1 *l.* payable if a life aged one year lives to 92, and, therefore, the value of the Life Annuity is proved to be true.—The value of 1 *l.* payable if a life of one year lives to be 90 is $\frac{4}{849}$ multiplied into 1 *l.* diſcounted for 89 years, (or .030481,) which is equal to .0001436. The value of a life aged 90 calculated from the value of a life aged 91 in the manner directed in page 58, is .7119. Theſe multiplied into each other will be equal to .0001022, which is alſo the ſum of the values of 1 *l.* payable if a life aged one year continues to the ſeveral ages of 91 and 92; conſequently the value of this Life-Annuity is alſo true.—Again, the ſum of the three values of 1 *l.* payable if a life of one year continues to the ſeveral ages of 90, 91, and 92 is .0002458. The value of a life aged 89 calculated from the value of a life aged 90 is 1.09737; and the value of 1 *l.* payable if a life aged one year continues to the age of 89 is $\frac{6}{849}$ multiplied into 1 *l.* diſcounted for 88 years (or into .0317) which is equal to .000224. And 1.09737 multiplied into .000224 is equal to .0002458, or the ſum of the values of 1 *l.* juſt found.—From hence it follows that 1.09737 is the true value of an Annuity of 1 *l.* on a life aged 89 years.—By proceeding in this manner through every age, it will appear at laſt that the ſum of all the values of 1 *l.* payable if a life aged one year continues to the ſeveral ages of 92, 91, 90, 89, 88,——and two years, is 13.719, which correſponds *exactly*, as may be ſeen in the following table, with the value of an Annuity of 1 *l.* on the ſame life found, in conſequence of computing upwards from the oldeſt to the youngeſt life, according to the theorem in page 57, and therefore the values of all the lives are ſtrictly true*.

In making theſe calculations it will be convenient to diſpoſe the operations into the following order.

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* See the demonſtration of this rule in note B. of the Appendix.

FIRST TABLE *of the Values of Single Live*

Age.	Values of Lives.	Values of 1 *l.* payable, &c.	Sums.
92	0000000	.0000332	0000000
91	.480769	.0000690	.0000332
90	.711908	.0001436	.0001022
89	1.097377	.0002240	.0002458
88	1.512531	.0003106	.0004698
87	1.93271	.0004038	.0007804
86	2.16983	.0005460	.0011842
85	2.33075	.0007425	.0017302
84	2.59261	.0009539	.0024727
83	2.79011	.0012283	.0034266
82	2.96102	.0015724	.0046549
81	3 12506	.0019929	.0062273
80	3.36282	.0024446	.0082202
79	3.64096	.0029293	.0106648
78	3.94186	.0034488	.0135941
77	4.25533	.0040052	.0170429
76	4.51419	.0046626	.0210481
75	4.79106	.0053666	.0257107
74	5.07880	.0061192	.0310773
73	5.37267	.0069235	.0371965
72	5.66943	.0077823	.0441200
71	5.96681	.0086987	.0519023
70	6.26315	.0096760	.0606010
69	6.55730	.0107175	.0702770
68	6.84841	.0118269	.0809945
67	7.13585	.0130079	.0928214
66	7.41916	.0142645	.1058293
65	7.69802	.0156007	.1200938
64	7.97220	.0170211	.1356945
63	8.24154	.0185300	.1527156
62	8.50594	.0201326	.1712456
61	8.76533	.0218336	.1913782

Age.	Values of Lives.	Values of 1 *l.* payable, &c.	Sums.
60	9.01970	.0236385	.2132118
59	9.26904	.0255529	.2368503
58	9.51338	.0275826	.2624032
57	9.75275	.0297338	.2899858
56	9.98721	.0320130	.3197196
55	10.21681	.0344269	.3517326
54	10.44163	.0369827	.3861595
53	10.66175	.0396879	.4231422
52	10.87724	.0425503	.4628301
51	11.08819	.0455782	.5053804
50	11.25491	.0489527	.5509586
49	11.42161	.0525242	.5999113
48	11.62641	.0561166	.6524355
47	11.82645	.0599124	.7085521
46	12.02188	.0639221	.7684645
45	12.21282	.0681567	.8323866
44	12.39942	.0726278	.9005433
43	12.58179	.0773475	.9731711
42	12.76005	.0823286	1.0505186
41	12.93432	.0875844	1.1328472
40	13.10472	.0931290	1.2204316
39	13.30703	.0987116	1.3135606
38	13.50266	.1045918	1.4122722
37	13.69198	.1107846	1.5168640
36	13.87528	.1173053	1.6276486
35	14.05285	.1241705	1.7449539
34	14.22945	.1313973	1.8691244
33	14.39185	.1390035	2.0005217
32	14.55378	.1470079	2.1395252
31	14.71095	.1554303	2.2865331
30	14.86358	.1642913	2.4419634
29	14,97797	.1740040	2.6062547
28	15.09091	.1842335	2.7802587
27	15.20236	.1950015	2.9644922
26	15.31231	.2063363	3.1594937
25	15.42074	.2182658	3.3658300

Age.	Values of Lives.	Values of 1 *l.* payable, &c.	Sums.
24	15.52765	.2308196	3.5840958
23	15.63303	.2440284	3.8149154
22	15.73689	.2579246	4.0589438
21	15.83922	.2725436	4.3168684
20	15.94004	.2879162	4.5894120
19	16.07007	.3035028	4.8773282
18	16.22701	.3192710	5.1808310
17	16.40904	.3351862	5.5001020
16	16.58389	.3518634	5.8352882
15	16.75190	.3693391	6.1871516
14	16.91339	.3876497	6.5564907
13	17.06868	.4068341	6.9441404
12	17.21805	.4269330	7.3509745
11	17.33100	.4487846	7.7779075
10	17.44042	.4717013	8.2266921
9	17.51606	.4965939	8.6983934
8	17.53046	.5245133	9.1949873
7	17.51879	.5548026	9.7195006
6	17.42642	.5895802	10.2743032
5	17.26414	.6292728	10.8638834
4	16.99078	.6764330	11.4931562
3	16.62953	.7318042	12.1695892
2	15.77754	.8177041	12.9013934
1	13.71908	1.0000000	13.7190975

In the firſt of theſe columns is the age.—In the ſecond, the value of a life at that age agreeable to the *Northampton* Table of Obſervations *, found by calculating upwards from the oldeſt to the youngeſt in the manner already directed. That is (reckoning intereſt at 4 *per cent.)* 000000 = $\frac{0}{1} \times \frac{1}{1.04}$ ---- .480769 = $\frac{1}{2} \times \frac{1}{1.04}$ ---- .711908 = $\frac{2}{4} \times \frac{1}{1.04} \times \overline{1 + .480769}$ ---- 1.09737 = $\frac{4}{6} \times \frac{1}{1.04} \times \overline{1 + .711908}$, &c. &c.

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* See Table 3d, Appendix.

In the third column is the value of 1 *l.* payable to a life aged one year (ſuppoſed the youngeſt life to which the operations are to be carried) provided it ſhould exiſt to the ages 92, 91, 90, 89, &c.—That is; .0000332 is the probability that a life aged one year ſhall exiſt till 92 multiplied by the value of 1 *l.* payable at the end of 91 years; or, .0000332 = $\frac{1}{849} \times .02818$.——In like manner; .000069 = $\frac{2}{849} \times .02931$ ---- .0001436 = $\frac{4}{849} \times .030481$ *, &c. &c.

In column 4th are the ſums of all the values in the third column *above* the values even with thoſe ſums; or (which comes to the ſame) the ſums of the two numbers immediately above them, and even with one another in column 3d and column 4th. That is; .0000332 = $\overline{.0000332 + .0000000}$ ---- .0001022 = $\overline{.0000690 + .0000332}$ ---- .0002458 = $\overline{.0001436 + .0001022}$, &c. &c.

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* See Table 1ſt, Appendix.

In conſequence of this arrangement it will be always found that the product of every number in the ſecond column multiplied by the number even with it in the third column gives the number even with both in the fourth column; and this forms a proof during the whole progreſs of the calculations, that there has been no error in any of the preceding calculations; and that the value of every life, as it comes out, may be depended on to as many places of decimals, as the decimal figures in the product juſt mentioned are the ſame with the decimal figures in the correſpondent number in column 4th.—For example. 1.097377 (the value of a life aged 89) multiplied by the number even with it in the next column, or by .000224, gives .0002458, which being the ſame to the laſt place of decimals with .0002458 in the fourth column, proves that all the preceding operations have been right, and that the values of the life aged 89 and of all the lives from the oldeſt to 89 may be depended on to the ſix places of decimals.

This proof makes a little more labour neceſſary in calculating the values of lives; but this addition of labour is more than compenſated by the time and trouble that muſt be ſaved in conſequence of detecting every error as it ariſes; and alſo by the ſatisfaction attending a ſeries of operations, every one of which, if right, proves the truth of all that have gone before it.—The following expedient will, however, remove one half of the labour.—In order to find the numbers in the third column one multiplication is neceſſary and one diviſion.—All the time employed in the latter may be ſaved by making the number 1000 the diviſor inſtead of the number oppoſite, in the Table of Obſervations, to the youngeſt age. For example. In all the operations which bring out the numbers in column 3d, 849 is the diviſor; but it might have been taken to be 1000; and then the firſt number in that column, inſtead of .0000332, would have been $.00002818 = \frac{1}{1000} \times .02818$. The ſecond number, inſtead of .0000690 would have been $.00005862 = \frac{2}{1000} \times .02931$. The third inſtead of .0001436 would have been $.00012192 = \frac{4}{1000} \times .030481$, &c. &c.—— And the correſpondent ſums in the fourth column would have been .0000000 ---- .00002818 ---- .00008680 ---- .00020872 ---- &c. &c.—The Table then would have been as follows.

SECOND TABLE *of the Values of Sing Lives.*

Age.	Values of Lives.	Values of 1 *l.* payable, &c.	Sums.
92	00000000	.00002818	.00000000
91	.480769	.00005862	.00002818
90	.711908	.00012192	.00008680
89	1.097377	.00019020	.0002087
88	1.512531	.00026376	.0003989
87	1.93271	.0003428	.0006626
86	2.16983	.0004635	.0010054
85	2.33075	.0006304	.0014689
84	2.59261	.0008099	.0020993
83	2.79011	.0010429	.0029092
82	2.96102	.0013349	.0039521
81	3.12506	.0016920	.0052870
80	3.36282	.0020755	.0069790
79	3.64096	.0024870	.0090545
78	3.94186	.0029281	.0115415
77	4.25533	.0034005	.0144696
76	4.51419	.0039588	.0178701
75	4.79106	.0045563	.0218289
74	5.07880	.0051953	.0263852
73	5.37267	.0058781	.0315805
72	5.66943	.0066072	.0374586
71	5.96681	.0073852	.0440658
70	6.26315	.0082149	.0514510
69	6.55730	.0090992	.0596659
68	6.84841	.0100411	.0687651
67	7.13585	.0110438	.0788062
66	7.41916	.0121106	.0898500
65	7.69802	.0132451	.1019606
64	7.97220	.0144509	.1152057
63	8.24154	.0157321	.1296566
62	8.50594	.0170926	.1453887

Age.	Values of Lives.	Values of 1 *l.* payable, &c.	Sums.
61	8.76533	.0185368	.1624813
60	9.01970	.0200692	.1810181
59	9.26904	.0216945	.2010873
58	9.51338	.0234177	.2227818
57	9.75275	.0252440	.2461995
56	9.98721	.0271791	.2714435
55	10.21681	.0292285	.2986226
54	10.44163	.0313983	.3278511
53	10.66175	.0336951	.3592494
52	10.87724	.0361253	.3929445
51	11.08819	.0386960	.4290698
50	11.25491	.0415609	.4677658
49	11.42161	.0445931	.5093267
48	11.62641	.0476430	.5539198
47	11.82645	.0508657	.6015628
46	12.02188	.0542699	.6524285
45	12.21282	.0578650	.7066984
44	12.39942	.0616610	.7645634
43	12.58173	.0656681	.8262244
42	12.76005	.0698970	.8918925
41	12.93432	.0743591	.9617895
40	13.10472	.0790665	1.0361486
39	13.30703	.0838062	1.1152151
38	13.50266	.0887985	1.1990213
37	13.69198	.0940560	1.2878198
36	13.87528	.0995923	1.3818758
35	14.05285	.1054208	1.4814681
34	14.22494	.1115563	1.5868889
33	14.39185	.1180139	1.6984452
32	14.55378	.1248097	1.8164591
31	14.71095	.1319604	1.9412688
30	14.86358	.1394834	2.0732292
29	14.97797	.1477305	2.2127126
28	15.09091	.1564143	2.3604431
27	15.20236	.1655564	2.5168574

Age.	Values of Lives.	Values of 1 *l.* payable, &c.	Sums.
26	15.31231	.1751795	2.6824138
25	15.42074	.1853077	2.8575933
24	15.52765	.1959658	3.0429010
23	15.63303	.2071801	3.2388668
22	15.73689	.2189780	3.4460469
21	15.83922	.2313882	3.6650249
20	15.94004	.2444409	3.8964151
19	16.07007	.2576738	4.1408560
18	16.22701	.2710611	4.3985298
17	16.40904	.2845731	4.6695909
16	16.58389	.2987323	4.9541640
15	16.75190	.3135689	5.2528963
14	16.91339	.3291146	5.5664652
13	17.06868	.3454022	5.8955798
12	17.21805	.3624662	6.2409820
11	17.33100	.3810182	6.6034482
10	17.44042	.4004744	6.9844664
9	17.51606	.4216082	7.3849408
8	17.53046	.4453118	7.8065490
7	17.51879	.4710274	8.2518608
6	17.42642	.5005436	8.7228882
5	17.26414	.5342526	9.2234318
4	16.99078	.5742916	9.7576844
3	16.62953	.6213018	10.3319760
2	15.77754	.6942307	10.9532778
1	11.64748	1.0000000	11.6475085

In this way the ſame proof of all the operations is obtained; nor has any error ariſen except in the value of the life laſt calculated, which is very eaſily corrected by taking the number of the living at the age of this life agreeable to the Table of Obſerations, inſtead of ſuppoſing it 1000: that s, by making the value of the life aged one ear $\frac{722}{849} \times \frac{1}{1.04} \times \overline{1 + 15.77754}$ inſtead of $\frac{722}{000} \times \frac{1}{1.04} \times \overline{1 + 15.77754}$.

The uſe of logarithms would further exedite the calculations; and by theſe means ſkilful perſon may, in three or four hours, alculate the values of ſingle lives at all ges, agreeably to any given table of obſerations, without the poſſibility of commiting any miſtakes *.

Having ſaid ſo much on the values of nnuities on *Single* Lives, the method of nding the values of Annuities on the *joint ntinuance* of any ſingle lives will be eaſily derſtood.——Call the value of any two int lives M, the probability that two lives e year younger will exiſt a year $\frac{bd}{ac}$ and $\frac{1}{r}$, before, the value of 1 *l.* payable at the d of the year. Then by reaſoning as in ge 57, the value of the joint continuance two lives one year younger will be exeſſed by $\frac{bd}{acr} \times \overline{1 + M}$.——This, together ith the neceſſary proof of the operations, will be ſufficiently explained by the following ſpecimens.

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* The two following Tables have been computed in is manner.

SPECIMEN *of the eaſieſt Method of computing the Value of any two Joint Lives.*

Let the two lives be equal; the rate of intereſt 4 *per cent.* or $r = 1.04$; and the table of obſervations the *Northampton* Table, given in the Appendix.

Ages.		Values of two equal joint lives.	Values of 1 *l.* payable, &c.	Sums.
92	92	.000000	.000000002818	.00000000000
91	91	.240383	.000000011723	.000000002818
90	90	.298169	.000000048768	.0000001454
89	89	.55477	.000001114120	.00000063309
88	88	.84092	.000002111008	.00000177429
87	87	1.13287	.0000034280	.00000388437
86	86	1.21351	.0000060255	.00000731237
85	85	1.24462	.0000107168	.00001333787
84	84	1.4143	.0000170079	.00002405467
83	83	1.5144	.0000271154	.0000410625
82	82	1.5998	.0000427168	.0000681779
81	8	1.6868	.0000659880	.0001108947
80	80	1.8570	.0000954730	.0001768827
79	79	2.0694	.0001318110	.0002723557
78	78	2.2976	.0001756860	.0004041667
77	77	2.5427	.0002278335	.0005798527
76	76	2.7182	.0002970100	.0008076862
75	75	2.9192	.0003781629	.0011046962
74	74	3.1364	.0004727723	.0014828591
73	73	3.3605	.0005819219	.0019556314
72	72	3.5893	.0007069704	.0025375533

Ages.		Values of two equal joint lives.	Values of 1 *l.* payable, &c.	Sums.
71	71	3·8203	.0008492980	.0032445237
70	70	4·0516	.0010104327	.0040938217
69	69	4·2821	.0011919952	.0051042544
68	68	4·5111	.0013957129	.0062962496
67	67	4·7381	.0016234386	.0076919625
66	66	4·9626	.001877143	.0093154011
65	65	5·1843	.002158951	.011192544
64	64	5·4031	.002472104	.013351495
63	63	5·6188	.002816055	.015823599
62	62	5·8313	.003196316	.018639654
61	61	6.0407	.003614776	.021835970
60	60	6.2468	.004074048	.025450746
59	59	6.4497	.004577540	.029524794
58	58	6.6494	.005128476	.034102334
57	57	6.8459	.005730388	.039230810
56	56	7·0392	.006387088	.044961198
55	55	7·2294	.007102526	.051348286
54	54	7·4165	.007880973	.058450812
53	53	7·6006	.008727031	.066331785
52	52	7·7818	.009645455	.075058816
51	51	7·9599	.01064140	.084704271
50	50	8.0779	.01180330	.09534567
49	49	8.2008	.01306578	.10714897
48	48	8.3829	.01434054	.12021475
47	47	8.5608	.01571750	·13455529
46	46	8.7350	.01720355	·15027279
45	45	8.9054	.01880613	.16747634
44	44	9 0723	.02053311	.18628247
43	43	9·2358	.02239282	.20681558
42	42	9·3960	.02439405	.22920840
41	41	9·5532	.02654620	.25360245
40	40	9·7065	.02885927	.28014865
39	39	9·9110	.03117591	·30900792
38	38	10.1073	.03365463	·34018383
37	37	10.2962	.03630562	.37383849

Ages.		Values of two equal joint lives.	Values of 1 *l.* payable, &c.	Sums.
36	36	10.4782	.03913977	.41014411
35	35	10.6538	.04216832	.44928388
34	34	10.8240	.04540341	.49145220
33	33	10.9880	.04885776	.53685561
32	32	11.1462	.05254488	.58571337
31	31	11.3007	.05647905	.63825825
30	30	11.4500	.06067528	.69473730
29	29	11.5422	.06544461	.75541258
28	28	11.6351	.07054285	.82085719
27	27	11.7292	.07599039	.89140004
26	26	11.8242	.08180883	.96739043
25	25	11.9191	.08802116	1.04919926
24	24	12.0139	.09465148	1.13722042
23	23	12.1089	.10107254	1.23187190
22	22	12.2038	.1092700	1.3335973
21	21	12.2985	.1173138	1.4428673
20	20	12.3928	.1258871	1.5601811
19	19	12.5346	.1345057	1.6860682
18	18	12.7201	.1431203	1.8205739
17	17	12.9461	.1516775	1.9636942
16	16	13.1616	.1607180	2.1153717
15	15	13.3673	.1702679	2.2760897
14	14	13.5638	.1803548	2.4463576
13	13	13.7516	.1910074	2.6267124
12	12	13.9311	.2022562	2.8177198
11	11	14.0530	.2148943	3.0199760
10	10	14.1709	.2282704	3.2348703
9	9	14.2356	.2432679	3.4631407
8	8	14.2031	.2609527	3.7064086
7	7	14.1320	.2807323	3.9673613
6	6	13.9355	.3048311	4.2480936
5	5	13.6352	.3339079	4.5529247
4	4	13.1723	.3709924	4.8868326
3	3	12.5931	.4175148	5.2578250
2	2	11.3227	.5012346	5.6753398
1	1	6.1765*	1.0000000	6.1765744

* The *true* value of theſe two joint lives is 10.5432.

In the third column the values are $\frac{0}{1} \times \frac{1}{1.04} = 0000000$ ----- $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{1.04}$* $\times \overline{1 + 0} = .240383$ ----- $\frac{2}{4} \times \frac{2}{4} \times \frac{1}{1.04} \times \overline{1 + .240383} = .298169$ ----- $\frac{4}{6} \times \frac{4}{6} \times \frac{1}{1.04} \times \overline{1 + .298169} = .554772$ ----- $\frac{6}{8} \times \frac{6}{8} \times \frac{1}{1.04} \times \overline{1 + .554772} = .84092$ --- and ſo on.

In the fourth column the firſt value is the probability that two lives aged one year will *both* exiſt to the age of 92 multiplied by the value of 1 *l.* payable at the end of 91 years; or (taking 1000 for the number living at one year of age) $\frac{1}{1000} \times \frac{1}{1000} \times .028182$† $= .000000028182$. ----- The ſecond value is $\frac{2}{1000} \times \frac{2}{1000} \times .029309 = .00000011723$; or $\frac{2}{1000}$ multiplied by 00005862 (the ſecond number in the third column of the ſecond Table of the values of *ſingle* lives, page 70.) Alſo the third value is $\frac{4}{1000}$ multiplied by .00012192, the third number in the third column of the ſame Table. The fourth value is $\frac{6}{1000}$ multiplied by .0001902, the fourth number in the third column of the ſame table; and ſo on.

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* Had $\frac{1}{2} \times \frac{1}{1.04}$, $\frac{2}{4} \times \frac{1}{1.04}$, $\frac{4}{6} \times \frac{1}{1.04}$, &c. been preſerved from the computation of ſingle lives in page 70, further ſaving of labour, perhaps, might have been produced.

† See Table 1ſt, Appendix.

In the fifth column are the ſums reſpectively of .0000002818 and 0000000; of .0000001 1723 and .0000002818, &c. —And the equality of the products of the values even with one another in columns third and fourth to the numbers in the ſame line in the fifth column forms the ſame proof of the accuracy of the operations with that already explained in giving an account of the method of computing the values of Single Lives.

SECOND SPECIMEN.

Let the two lives be unequal, the difference of age 60 years, and the rate of intereſt and table of obſervations the ſame.

Ages.		Values of two joint Lives.	Values of 1 *l.* payable, &c.	Sums.
92	32	.0000000	.00012481	00000000
91	31	.472906	.00026392	.00012481
90	30	.69673	.00055793	.00038873
89	29	1.0680	.00088638	.00094666
88	28	1.4649	.00125131	.00183304
87	27	1.8630	.00165556	.00308435
86	26	2.0813	.00227733	.00473991

Ages.		Values of two joint Lives.	Values of 1*l.* payable, &c.	Sums.
85	25	2.2275	.00315023	.00701724
84	24	2.4706	.00411528	.01016747
83	23	2.6515	.00538668	.01428275
82	22	2.8070	.00700730	.01966943
81	21	2.9561	.00902414	.02667673
80	20	3.1750	.01124428	.03570087
79	19	3.4375	.01365671	.04694515
78	18	3.7262	.01626367	.06060186
77	17	4.0315	.01906640	.07686553
76	16	4.2818	.02240492	.09593193
75	15	4.5469	.02602622	.11833685
74	14	4.8203	.02994943	.14436307
73	13	5.0977	.03419482	.17431250
72	12	5.3762	.03878388	.20850732
71	11	5.6437	.04381709	.24729120
70	10	5.9098	.04925835	.29110829
69	9	6.1626	.05523067	.34036664
68	8	6.3911	.06189834	.39559731
67	7	6.6073	.06924103	.45749565
66	6	6.7891	.07758426	.52673668
65	5	6.9396	.08708317	.60432094
64	4	7.0405	.09820386	.69140411
63	3	7.1000	.11121302	.78960797
62	2	6.9390	.12982114	.90082099
61	1	1.0306*	1.00000000	1.03064213

Here in the third column 00000 = $\frac{179}{187} \times$
$\frac{0}{1} \times \frac{1}{1.04}$ ---- .472906 = $\frac{187}{195} \times \frac{1}{2} \times \frac{1}{1.04} \times$
1 + 0 ---- ,69673 = $\frac{195}{203} \times \frac{2}{4} \times \frac{1}{1.04} \times$

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* The *true* value of theſe two joint lives is 6.2254.

$\overline{1+.472906}$ - - - $1.068 = \frac{203}{211} \times \frac{4}{6} \times \frac{1}{1.04} \times \overline{1 + .69673}$, and ſo on.

In the fourth column $.00012481 = .12481 \times \frac{1}{1000}$ - - - - $.00026392 = .1319604 \times \frac{2}{1000}$ - - - - $.00055793 = .1394834 \times \frac{4}{1000}$, and ſo on.

In computing this fourth column, the number of the living at each of the youngeſt ages to which the values are to be found (that is, at the ages 1 and 61) are ſuppoſed to be 1000; and it appears in both theſe examples that half the trouble of computing the numbers in this column is ſaved by having previouſly computed the Table (in page 70) of the values of ſingle lives, and finding there the numbers .12481—.1319604—.1394834 oppoſite to the ages 32, 31,—30, &c. in the ſecond ſpecimen, and the numbers .00002818, .00005862, .00012192 oppoſite to the ages 92, 91, 90, &c. in the firſt ſpecimen. The proof in all theſe examples is the ſame.

The calculation of the values of joint lives in exact agreement to a given table of obſervations has been conſidered as a taſk ſo great that it has been ſeldom undertaken. —We have therefore hitherto been obliged to ſatisfy ourſelves (as I have before obſerved, page 50) with the values of joint lives calculated by an eaſy rule given in the next chapter, Problem 3d, founded on Mr. *De Moivre*'s hypotheſis of an equal decrement of human life through all its ſtages; but this hypotheſis not correſponding ſufficiently to fact, the values of joint lives derived from it are not ſufficiently correct; and when combined (as they muſt be in computing the values of Reverſions and of Annuities dependent on Survivorſhips) with other inaccuracies derived from the ſame hypotheſis, produce errors in the ſolution of many queſtions which are too conſiderable.—The excellent Mr. *Simpſon* has indeed given a Table of the values of two joint lives agreeable to the London Table of Obſervations; but this table of obſervations ought not to be much uſed, becauſe, repreſenting the rate of mortality among the inhabitants of London taken in the groſs, it gives the values of lives much too low for the middling and the better ſort of people in London itſelf.—It is neceſſary I ſhould add, that Mr. *Simpſon*'s Tables of the values of lives in London give them only to one place of decimals, whereas it is in many caſes abſolutely neceſſary that they ſhould be given to at leaſt two or three places of decimals.—I have, therefore, in the preceding pages been more minute than I ſhould otherwiſe have been in deſcribing the ſafeſt and moſt expeditious methods of calculating ſuch values.

And, as by theſe methods the calculations are rendered pleaſant as well as expeditious, I hope that ere long ſome perſon will undertake them, chuſing for his guide the *Northampton* Table of Obſervations, which perhaps is better fitted for common uſe than any other.

CHAP. III.

An Account of the Method of Computing in all Caſes the Values of Aſſurances, and of Reverſions depending on any one, two, or three Lives, or any Survivorſhips between them.

INTRODUCTION.

IN the two preceding chapters I have en-
deavoured to explain the principles from
which the values of Aſſurances and Annui-
ies are deduced, without entering into the
olution of any caſes in particular, but re-
erring occaſionally to the writings of thoſe
athematicians where they may be found
n their greateſt accuracy.—The Problems
n this chapter are connected no further
han any other Problems with the obſerva-
ions in the preceding chapters; all of them
eing determined from the ſame common
rinciples.

The eighteenth of the following Problems is the ſame with the 38th in Mr. *Simpſon*'s Select Exerciſes; and from the ſolution of this Problem, the anſwers to the 19th, 20th, 21ſt, 22d, 24th, 25th, and 26th are partly derived.

The 23d, 27th, 28th, and all the ſubſequent Problems for determining the values of Reverſions depending on Survivorſhips between three lives for their whole duration and for terms, are no ways deduced from the ſolution of Mr. *Simpſon*'s Problem. Theſe, together with the Problems juſt mentioned, comprehend, as far as I can perceive, all the different caſes of queſtions in which any Survivorſhip between three lives can be involved.

The 4th, 5th, 6th, 7th, 8th, 9th, 14th, and 15th Problems (the 4th, 8th, and 9th taken from Dr. *Price*, and the reſt from Mr. *Simpſon)* have been chiefly inſerted as neceſſary to the ſolution of thoſe that follow: And, in order to render the work more complete, I have added *Corollaries* to ſome of theſe Problems of Dr. *Price* and Mr. *Simpſon*, from the ſolutions of which and their reſpective Problems, the values of moſt Aſſurances may be determined wherein one or two lives are concerned.

The 10th, 12th, 14th, and 15th Problems contain the ſolutions of caſes which have lately offered themſelves in practice, and, therefore, are not improperly introduced here.—In ſhort, the greateſt part of this chapter is new. Moſt of the Problems have never been ſolved before, eſpecially thoſe which determine the values of Reverſions, depending on Survivorſhips between three lives, *for terms.*—Mr. *Dodſon* in his *Mathematical Repoſitory* has, indeed, given the ſolutions of ſome of them for the *whole duration of life* and in *one preſent* payment; but theſe ſolutions are rendered in a great meaſure uſeleſs, not only as they are given in *algebraical ſymbols*, which diſcourage the generality of readers who are not acquainted with the mathematics, but alſo as they are derived from an hypotheſis which produces very conſiderable errors when more than two lives are involved in the ſurvivorſhip. I have therefore, endeavoured to give thoſe ſolutions, ſo as to be adapted to any rate of mortality; and, to the Rules for determining the values in *ſingle* payments, I have alſo added other Rules for determining the ſame values in *annual* payments.

With regard to the demonſtrations of the following Problems; it is evident, that all thoſe which derive their ſolutions from Dr. *Price*'s and Mr. *Simpſon*'s Problems require no explanation.——Such, however, as are not connected with any of the fore-mentioned Problems, and whoſe ſolutions are conſequently not ſo obvious, I have demonſtrated in the *Appendix*; thinking it beſt, in order to avoid deterring readers who are not mathematicians, to exclude all *analytical operations* from the body of the work.

The Tables in the Appendix have been inſerted merely for the ſake of illuſtrating the Problems by examples.—On this account they are confined to one rate of intereſt; but if the calculations are required at a higher or lower rate, other Tables will be found in moſt books on the ſubject; and the application of them being underſtood in one caſe, will eaſily be underſtood in every other.

In the foregoing chapter I have given an account of the moſt expeditious and eaſy method of computing tables of the values of ſingle and joint lives in exact agreement to the decrements of life or rate of mortality in any particular place. The labour of computing ſuch tables has been hitherto reckoned ſo great, that (except by Mr. *Simpſon* for *London*) it has never been undertaken.

taken.—Having then no ſuch tables, except for the very worſt lives and carried only to one place of decimals, I have been the more deſirous, in order a little to ſupply this defect, to compute thoſe in the preceding ſection.—But theſe computations having been meant chiefly as *ſpecimens*, and confined to one rate of intereſt and to the *Northampton* Table of Obſervations, it becomes neceſſary, in order to prepare the way for what is to follow in this chapter, to explain an eaſy method of *approximating* to the value of any life or lives according to any given rate of mortality. With this view I ſhall introduce the three firſt Problems, whoſe ſolutions afford the beſt and the only means, I am acquainted with, for ſupplying the want of the *real* values with any tolerable exactneſs.

PROBLEM I.

To find the *expectation* of any given life according to any given Table of the decrements of life.

SOLUTION.

Find the ſum of all the living in the Table from the age of the given life. Divide this ſum by the number of the living at that age; and the quotient leſſened by half unity will be the required expectation.

EXAMPLE.

The ſum of all the living from the age of ten in the *Northampton* Table of Obſervations (ſee Appendix, Table 3d) is 22811. This ſum divided by 570, the number living at the age of ten, gives 40.02, which leſſened by *half*-unity gives 39.52, or the expectation of a life of the age of ten according to the *Northampton* Table of Obſervations.

The *expectation* of a life is the ſame with its value ſuppoſing money not to bear intereſt; and in *De Moivre*'s *hypotheſis*, it is always *half the difference* between the age of a life and 86.

The *expectation* of a life at a given age, is alſo the ſame with the *time* or *number of years* which one with another a body of people at that age will live according to a given table of obſervations, ſome living as much *longer* as others live *leſs* than that time,

PROBLEM II.

To approximate to the value of a given life according to any given table of obſervations.

SOLUTION.

Find in Table 2d the value of an Annuity certain for a number of years equal to the *complement*, that is, to double the *expectation* of the life. Multiply this value by the perpetuity increaſed by unity, and divide the product by the complement.—The *quotient* ſubtracted from the perpetuity will be the value.

EXAMPLE.

The expectation of a life aged ten, by Table 4th, is 39.52. Double this expectation, or the *complement* of the life, is 79.04. ——The value of an Annuity certain for 79.04 years is (by Table 2d) 23.874. The product of 23.874 into 26 (or the perpetuity increaſed by unity) is 620.724; which divided by 79.04 (the complement) gives 7.85. And this quotient ſubtracted from 25 (the perpetuity) gives 17.15 years purchaſe, the value *nearly* of a life aged ten according to the *Northampton* Table of Obſervations.——The ſame value as calculated in exact agreement to the *Northampton* Table is (by Table, page 64) 17.44.

The value found by the ſame rule of a life aged ten according to the *London* Table of Obſervations * is 16.27 years purchaſe; and this value as given by Mr. *Simpſon* in exact agreement to the *London* Table is 16.4.

The value found by the rule of a life aged twenty, according to the *Northampton* Table of Obſervations, is 15.93 years purchaſe; and of a life aged thirty, 14.79.—Theſe values calculated in *exact* agreement to the *Northampton* Table are (by the Table, page 64) 15.94 and 14.86.

SCHOLIUM.

This rule gives the values of lives on the hypotheſis of an equal decreaſe in the probabilities of living; and it is by a rule, which agrees with this, that Mr. *De Moivre* has calculated the table commonly uſed of the values of lives according to his *hypotheſis*. —By this table the approximation required by the foregoing Problem may be found more compendiouſly in the following manner. "Take the difference between the "complement of the life and 86; and the "value deduced from Mr. *De Moivre*'s "Table correſponding to that difference, "provided it is not leſs than ten, will be "the value of the life."

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* See *Simpſon*'s Select Exerciſes, Table 2d and 5th, page 255 and page 260.

Thus; the complement of a life aged 15 being 73 by the *Northampton* Table *, the difference between it and 86 is 13, and the value correſponding to 13 in Mr. *De Moivre*'s *Table* of the values of lives †, is 16.60 reckoning intereſt at 4 *per cent.*; and this is nearly the value of a life aged 15 by the *Northampton* Table ‡.

PROBLEM III.

To approximate to the value of two joint lives, according to any given table of obſervations.

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* See Table 4th, Appendix.

† See Table 5th, Appendix.

‡ See Table, page 64.

SOLUTION.

Find the complements of the two lives, according to the table, by Problem 1ſt.—Find the *difference* between the complement of the *youngeſt* life and the complement of the *oldeſt* life increaſed by unity and twice the perpetuity. Multiply this difference by the value of an Annuity certain for a time equal to the complement of the *oldeſt* life, and by this complement divide the product, reſerving the quotient.

From twice the perpetuity ſubtract the reſerved quotient, and multiply the remainder by the perpetuity increaſed by unity *. This laſt product divided by the complement of the *youngeſt* life, and then ſubtracted from the perpetuity, will be the required value †.

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* N. B. When the complement of the youngeſt life is *greater* than the complement of the oldeſt life increaſed by unity and twice the perpetuity, the reſerved quotient, inſtead of being *ſubtracted* from twice the perpetuity, muſt be *added* to it, and the *ſum*, not the *difference*, multiplied by the perpetuity increaſed by unity. —If they are *equal*, the value of the joint lives will be "the perpetuity diminiſhed by twice the *quotient* ariſing "from dividing the perpetuity with its ſquare added, by "the common complement of the lives."

† This rule is the ſame with that given by Dr. *Price*, from Mr. *Simpſon*, in his Treatiſe on Reverſionary Payments, page 309, 3d edition.

EXAM-

EXAMPLE.

Let the joint lives propoſed be 10 and 15; and let the table of obſervations be Mr. *Simpſon's* for *London,* and the rate of intereſt 4 *per cent.*—The complements of the two lives are 69.6 and 63.8 (ſee Mr. *Simpſon's* Select Exerciſes, Table 2d, page 255.) The complement of the oldeſt life, increaſed by unity and twice the perpetuity, is 114.8, which leſſened by 69.6 (the complement of the *youngeſt* life) leaves 45.2 for the *reſerved remainder.*—This remainder multiplied by 22.95 (the value of an Annuity certain for 63.8 years by Table 2d, Appendix,) and the product divided by 63.8 (the complement of the *oldeſt* life) gives 16.26, the quotient to be reſerved; which ſubtracted from double the perpetuity, and the remainder (or 33.74) multiplied by the perpetuity increaſed by unity (or by 26) gives 877.24, which divided by 69.6 (the complement of the *youngeſt* life) and the quotient ſubtracted from the perpetuity, we have 12.40 for the value nearly of two joint lives aged 10 and 15 in London.——The exact value (by Mr. *Simpſon's* Table, Select Exerciſes, page 260) is 12.7.

The value found in the ſame way of two joint lives both 30, according to the decrements of life at Northampton, is 11.28; of two lives both 35 is 10.58. The exact values by the Table, page 74, are, 11.45 in the former caſe, and 10.65 in the latter. —If the ages are 30 and 40 the value of the joint lives (by this rule and the *Northampton* Table of Obſervations) will be 10.44. —If the ages are 46 and 56 the value will be 7.86.

What is ſaid in the Scholium to the laſt Problem muſt be applied to this.—And it ſhould be remembered further, that theſe approximations are to be uſed only when the decrements of life in a place are given, but no recourſe can be had to any tables giving the values of lives agreeably to thoſe decrements. This, as I have before obſerved, is at preſent almoſt univerſally the caſe; and the approximations now explained are, as far as I know, the only means that can be uſed to ſupply the defect.——Sometimes they deviate too much from exactneſs; but ſeldom more than in the examples I have given.

PROBLEM IV.

To determine the value of an Annuity on a given life for any number of years.

SOLUTION.

Find the value of a life as many years older than the given life as are equal to the term for which the Annuity is propoſed. Multiply this value by 1 *l.* payable at the end of this term, and alſo by the probability * that the life will continue ſo long.

Sub-

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* The probability that a given life ſhall continue any number of years or attain to a *given age*, is the fraction, whoſe *numerator* is the number of the living in any table of obſervations oppoſite to the given age, and *denominator* the number oppoſite to the preſent age of the given life. Thus; 203 is the number, in Table 3d, Appendix, oppoſite to 60, and 435 the number oppoſite to 30—— $\frac{203}{435}$ is, therefore, the probability that a perſon, whoſe age is 30, ſhall attain to 60, or live 30 years.

In the ſame manner the probability that any two lives ſhall continue in being together any number of years, or *both* attain to *given ages*, is the fraction, whoſe *numerator* is the product ariſing from the numbers of living, in the Table of Obſervations oppoſite to both thoſe *given ages*, multiplied by one another, and *denominator*, the product ariſing from the numbers oppoſite to the *preſent ages* multiplied into each other.—Hence the probability that two

lives

Subtract the product from the preſent value of the given life, and the remainder multiplied by the Annuity will be the anſwer.

lives aged 20 and 25 ſhall *both* continue in being 10 years is (by Table 3d) equal to $\frac{435\times400}{515\times475}=\frac{174000}{244625}$——Again: The probability that a life ſhall *fail* in any number of years, or *not* attain to a *given age*, is the fraction, expreſſing the chance of his *living* ſo long, ſubtracted from unity, agreeable to what has been obſerved in page 52: In other words; it is the fraction whoſe *denominator* is the number of the living oppoſite to the *preſent age*, and whoſe *numerator* is the *difference* between this *denominator* and the number of living at the *given age*. Thus; $\frac{203}{435}$ is the probability that a life of 30 ſhall attain to 60—— $\frac{232}{435}$, therefore, will be the probability that he ſhall *not* attain to this age.—For the ſame reaſon $\frac{435}{515}$ and $\frac{400}{475}$ being the probabilities that two lives aged 20 and 25 ſhall ſeverally *exiſt* 10 years; $\frac{80}{515}$ multiplied into $\frac{75}{475}$ $\left(=\frac{6000}{244625}\right)$ will be the probability that they ſhall *both die* in 10 years.—And this fraction again being ſubtracted from unity will give $\frac{238625}{244625}$ for the probability that they ſhall *not both* die in this time; that is, that one of them at leaſt ſhall *live* 10 years.

Further— $\frac{174000}{244625}$ is the probability that the lives of 20 and 25 ſhall exiſt together 10 years, which being ſubtracted from unity, or the *numerator* ſubtracted from the *denominator*, we ſhall have $\frac{70625}{244625}$ for the probability that they ſhall *not both* live ſo long; that is, that one of them at leaſt ſhall *die* in this time.

EXAMPLE.

Let the Annuity be 10 *l.* The age of the given life 30 years, and the term propoſed 15 years.—The value of a life aged 45 (or 15 years older than the given life) appears, by the Table in page 64 to be 12.2128. The value of 1 *l.* payable at the end of 15 years (by Table 1ſt, Appendix) is .5553; and the probability that the life will exiſt ſo long (by Table 3d) $\frac{325}{435}$. Theſe three quantities multiplied into each other are equal to 5.0668, which being ſubtracted from 14.8635, (the preſent value of the given life by the Table in page 64) we have 9.7967, and this remainder multiplied again into 10, gives 97.967 *l.* for the value required.

COROLLARY FIRST.

In the ſame manner may the value of an Annuity for any given term upon two *joint* lives be determined.—Thus; ſuppoſing the age of theſe two perſons to be 30 years, and the term propoſed 15 years.—The value of two joint lives aged 45 is (by Table, page 74) 8.9054, which multiplied into .5553 (the value of 1 *l.* at the end of 15 years by

Table 1ſt, Appendix) and alſo into $\frac{325\times325}{435\times435}$ (the probability * by Table 3d that *both* lives ſhall exiſt ſo long) will be equal to 2.7604, and this ſubtracted from 11.45 (the whole preſent value of the joint lives by the Table in page 74) leaves 8.6896 for their value during the term propoſed.——And ſince the value of the *longeſt* of two lives is equal to the *joint* lives ſubtracted from the ſum of the two *ſingle* lives †; if from twice 9.7967 we ſubtract 8.6896, we ſhall have alſo 10.9038 for the value of the longeſt of two equal lives aged 30 for the term juſt mentioned.——This rule may be applied to any ages or terms, and is further explained in Dr. *Price*'s Treatiſe on Reverſionary Payments, Schol. to Queſt. 6th.

COROLLARY SECOND.

The value of an Annuity for the remainder of a term certain after any life or lives is very eaſily obtained from the foregoing Problem and Corollary; the rule being as follows. " Find the value of an Annuity " on the life or lives for the given term by " the Problem or Corollary. Find alſo the

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* See the preceding note.

† See the Rule in page 55.

" value of an Annuity certain for the given " term by Table 2d. Subtract the former " of theſe two values from the latter, and " the remainder will be the anſwer." Thus; ſuppoſing the age of the given life to be 30 and the term 15 years. The value of the ſingle life for 15 years is found by the Problem to be 9.7967; and the value of an Annuity certain for this term is 11.1183. Therefore 1.3216 is the number of years purchaſe required.—In the ſame manner the value of an Annuity certain for the remainder of 15 years after the extinction of two *joint* lives aged 30 may be found by the aſſiſtance of the preceding Corollary to be equal to 2.4287; and for the remainder of the ſame term after the *longeſt* of two ſuch lives to be equal to .2145.

REMARK. It ſhould be obſerved that unleſs the value of the Annuity on the whole life is calculated to three places of decimals, as in the Table in page 64, thoſe rules will be defective, eſpecially if the term does not exceed ten years; in which caſe it will become neceſſary to compute the value of the Annuity in the Problem and firſt Corollary for each year ſeparately, agreeable to the directions in page 43. But if the Life Annuity, as in the preſent examples, is given in the Table to three or more places of decimals, the preceding ſolutions are ſufficiently correct in all caſes whatever.

PROBLEM V.

To find the value of an Annuity for three joint lives A, B, and C *.

SOLUTION.

Let A be the youngeſt and C the oldeſt of the three propoſed lives. Take the value of the two joint lives B and C, and find the age of a ſingle life D of the ſame value; then find the value of the joint lives A and D; which will be the anſwer.

EXAMPLE.

Let the three given ages be 20, 30, and 60.—The value of the two oldeſt joint lives B and C (by Table 6th) is 7.779, anſwering

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[*] A general account of the method of finding the values of *two* joint lives has been already given in page 48, &c. And Tables of theſe values for *London* have been given by Mr. *Simpſon* in his *Select Exerciſes*; and for general uſe agreeable to Mr. *De Moivre*'s hypotheſis by Dr. *Price* in his Treatiſe on Reverſionary Payments. This laſt Table I have inſerted in the Appendix.

ing to a ſingle life D of 65 years (by Table 5th,) and the value of the joint lives A and D (by Table 6th) is 6.882, which will be the value required.

If the values of the joint lives and the ſingle life had been taken as Mr. *Simpſon* has given them for *London*, this value would have been 5.82.

PROBLEM VI.

To find the value of an Annuity upon the longeſt of three lives, A, B, and C.

SOLUTION.

Let A be the youngeſt and C the oldeſt of the three propoſed lives. Find the value of the joint lives B and C, and find the age of a ſingle life D of the ſame value.—— Moreover, find by the rule in page 55 the value of the longeſt of the lives A and B, alſo that of the longeſt of the lives A and C, and likewiſe that of the longeſt of the lives A and D; then the laſt of theſe three values ſubtracted from the ſum of the two former leaves the value ſought.

EXAMPLE.

Let the three given ages be 20, 30, and 60.—The value of the oldeſt joint lives B and C (by Table 6th) is 7.779 anſwering to a ſingle life D (in Table 5th) of 65 years. The value of the longeſt of the two lives A and B, by the ſame Tables, is 18.864, alſo that of the longeſt of the two lives A and C is 16.941, and that of the longeſt of the two lives A and D is 16.64; therefore 19.165 is the value required.

Had the values of the joint lives and the ſingle life been computed from the *Northampton* Table of Obſervations by the rules in Problem 2d and 3d, this value would have been 19.005.

SCHOLIUM.

The *exact* value of an Annuity on the longeſt of three lives is found by "ſubtracting the values of all the joint lives, "combined *two and two*, from the ſum of "all the *ſingle* lives added to the value of "the *three joint* lives." Thus; by the Table in page 74 the value of two joint lives aged 60 is 6.2468, which multiplied into 3 becomes equal to 18.7404. By the Table in page 64 the value of a ſingle life aged 60 is 9.0197, which multiplied alſo into 3 becomes equal to 27.0591. And, by Table 7th Appendix, the value of *three* joint lives aged 60 is 4.7826, which being added to 27.0591, and 18.7404 ſubtracted from the *ſum*, we have 13.1013 for the *exact* value of the longeſt of three equal lives aged 60 by the *Northampton* Table of Obſervations. This value, by the rule in the preceding Problem and Mr. *De Moivre*'s hypotheſis, is 12.987.—In the ſame manner the value of three lives aged 70 by the rule in this Scholium and the *Northampton* Table of Obſervations may be found equal to 9.6126; and of three lives aged 75 equal to 7.6792.—By the rule in the Problem and Mr. *De Moivre*'s hypotheſis thoſe values are 8.969 and 6.537 reſpectively.——From hence it appears that the *hypotheſis* ſhould by no means be uſed in finding thoſe values after the age of 60. But with reſpect to the *approximation* in Problem 6th we muſt content ourſelves with it till the values of *three* joint lives are computed from real obſervations. This, however, is a work of ſuch immenſe labour that it is not likely ever to be performed; nor would it indeed be very neceſſary if we had only Tables of the values of *two* joint lives computed from thoſe obſervations; in which caſe the values of the longeſt of three lives might be always obtained ſufficiently near the truth from the approximation juſt mentioned. The want of ſuch tables is in a great meaſure ſupplied by the rules in Problem 2d and 3d; the values according to theſe differing ſo little from the real ones, that we ſhall ſeldom incur any material error in having recourſe to them. Thus in the preſent inſtances, the value of the longeſt of three lives aged 70, taking the complements of life from the *Northampton* Table, is equal to 9.46, and of three lives aged 75 equal to 7.586; both of which are only about $\frac{1}{10}$th of a year's purchaſe leſs than the *exact* values found by the ſame table.——Had the ages been younger this difference would have been ſtill more inconſiderable.

PROBLEM VII.

To find the value of an Annuity granted upon three lives A, B, and C, on condition of its ceaſing as ſoon as any two of them become extinct.

SOLUTION.

Find the value of each *pair* of joint lives, *viz.* of A and B, of A and C, and of B and C; then from the *ſum* of thoſe three values let twice the value of the three joint lives A, B, and C, be deducted, and the remainder will be the anſwer.

EXAMPLE.

Let the ages of A, B, and C reſpectively be 20, 30, and 60. By Table 6th the value of the joint lives

$$\left.\begin{array}{l}A,\ B\\ A,\ C\\ B,\ C\end{array}\right\}\text{ will be }\left\{\begin{array}{r}11.711\\ 7.967\\ 7.779\end{array}\right.$$

the *ſum* of which three numbers is 27.457. Moreover the value of the three joint lives A, B, and C, by Problem 5th is 6.882; therefore 13.693 is the value required.

Had the age of all the three lives been 60, the *exact* value by the *Northampton* Table of Obſervations would have been 9.175; and the ſame value by Mr. *De Moivre*'s hypotheſis is 8.99.

COROL-

COROLLARY.

The value of an Annuity for any given term upon the continuance of two lives out of three is determined much in the ſame manner; all that is neceſſary, being only to ſubſtitute in the preceding rule the value of the joint lives for the *given term*, inſtead of the values for *their whole duration.*

EXAMPLE.

Let the value of an Annuity be required upon three lives A, B, and C, aged 10, 10, and 70, for 15 years, on condition of its ceaſing if two of them ſhould fail in this time.——The value of two joint lives aged 10 for 15 years (by the Table in page 74 and Corollary to Problem 4th) is 9.578. The value of two joint lives aged 10 and 70 for 15 years, by the ſame Corollary, and the Table in page 78, is 5.768. The value of the three joint lives by Problem 5th and Corollary to Problem 4th, and alſo by the Tables in page 64, and page 78, is 5.53. Therefore the value of two out of the three given lives for 15 years is 10.054.

PROBLEM VIII.

To find the value of a given ſum payable at the deceaſe of A, *whenſoever* that ſhall happen. Or in other words; to find the value of an *Aſſurance* of any given ſum on the *whole duration* of the life of A.

SOLUTION.

Subtract the value of the life from the perpetuity. Multiply the remainder by the product of the given ſum into the intereſt of 100*l.* for a year: and this laſt product divided by 100*l.* increaſed by its intereſt for a year, will give the anſwer in a *ſingle preſent* payment.——And this payment divided by the value of the life will give the anſwer in *annual* payments during the continuance of life.

EXAMPLE.

Let the age of the life be 20.—The ſum 1000*l.*—By the table in page 64 the value of the life is 15.94, which ſubtracted from 25 (the perpetuity) leaves 9.06, and this remainder multiplied by the product of 1000*l.* into 4 or by 4000*l.* gives 36240, which being divided by 104 quotes 348.461*l.*

for the value in *one preſent* payment.—And this payment divided by 15.94 is 21.86 *l.* the ſame value in *annual* payments during the continuance of life.

The value of this Aſſurance by Mr. *De Moivre*'s hypotheſis is 350.346 *l.* in *one* payment, and 22.046 *l.* in *annual* payments.

COROLLARY.

The values of Aſſurances on any two or three *joint* lives, or on the *longeſt* of two or three, or two out of three lives are determined much in the ſame manner, only ſubſtituting the values of the *joint* lives, the longeſt of the lives, or of two out of three lives, inſtead of the value of the *ſingle* life in the preceding ſolution.—For example: Let the value be required in *preſent* money, and alſo in *annual* payments of 1000 *l.* to be received on the failure of the *firſt* of three lives whoſe ages are 20, 30, and 60.—By Problem 5th the value of three joint lives (according to Mr. *De Moivre*'s hypotheſis) is 6.882, which ſubtracted from 25 and multiplied into 4000 gives 72472, and this product divided by 104 quotes 696.84 *l.* for the anſwer in *one preſent* payment.—And this payment divided by 6.882 gives 101.25 *l.*

for the ſame value in *annual* payments during the joint continuance of the three lives.

Again. Suppoſe the value of 1000*l.* is required as above to be received on the failure of the *laſt* of three lives whoſe ages are 20, 30, and 60.—By Problem 6th the value of an Annuity on the longeſt of three lives (according to Mr. *De Moivre*'s hypotheſis) is 19.165. This ſubtracted from 25 (the perpetuity) leaves 5.835, which multiplied by 4000 and then divided by 104 gives 224.42*l.* for the value of the Aſſurance in *one preſent* payment.—And 224.42*l.* divided by 19.165 quotes 11.709*l.* for the ſame value in *annual* payments during the continuance of the longeſt of the three lives.

Laſtly. Suppoſe the value is required of 1000*l.* to be paid on the extinction of two lives out of three, whoſe ages are 20, 30, and 60.——By Problem 7th and Mr. *De Moivre*'s hypotheſis the value of an Annuity on the continuance of two out of the three given lives is 13.693. This ſubtracted from 25 (the perpetuity) leaves 11.307, which multiplied into 4000 and then divided by 104 quotes 434.884*l.* for the value in *one preſent* payment. And 434.884 divided by 13.693 gives 31.759 for the ſame value in *annual* payments during the continuance of two out of the three lives; or, which is the ſame thing, till the claim becomes due.

PROBLEM IX.

To find the value of a given ſum payable at the deceaſe of A, ſhould that happen within a given term. Or in other words; to find the value of an *Aſſurance* of any given ſum on the life of A for any given term.

SOLUTION.

From the value of an Annuity certain for the given term, found by Table 2d, ſubtract the value of the life for that term, found by Problem 4th, and *reſerve* the remainder. Multiply the value of 1 *l.* due at the end of the given term (found by Table 1ſt) by the perpetuity and alſo by the probability that the life of A ſhall fail in the given term; let the *product* be added to the *reſerved* remainder and multiplied by the given ſum; then will this laſt product, divided by the perpetuity increaſed by unity, quote the value in *one preſent* payment.—And if this payment be divided by the value of the life for the given term, we ſhall have the ſame value in *annual* payments.

EXAMPLE.

Let the age of A be 30. The ſum to be aſſured 1000 *l.* The term 15 years.—The value of the life of A for 15 years is, by Problem 4th and the *Northampton* Table, 9.7967, which ſubtracted from 11.1183 (the value of an Annuity certain for 15 years) leaves 1,3216 the remainder to be *reſerved.*—The value of 1 *l.* to be received at the end of 15 years is .5553. The probability that the life of A ſhall fail in 15 years is (by Table 3d) $\frac{110}{435}$ *; and the perpetuity is 25. Theſe numbers multiplied by one another, and 1.3216 added to the product, make 4.8319, which multiplied into 1000 (the given ſum) and divided by 26 (the perpetuity increaſed by unity) gives 185.842 *l.* the value required in a *ſingle preſent* payment.—And 185.842 *l.* divided by 9.7967 (the value of the life of A for the given term) quotes 18.969 *l.* the ſame value in *annual* payments for 15 years † ſubject to failure ſhould the life fail.

COROL-

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* See note, page 95.

† The payments in this rule are ſuppoſed to be made at the *end* of every year. But the general mode of making Aſſurances of this kind (and indeed of all others) is, by

COROLLARY.

In a ſimilar manner may the price of Aſſurance on any two or three *joint* lives, or on the *longeſt* of two or three, or two out of three lives, for any given term be calculated, only ſubſtituting the value of the *joint* lives, of the *longeſt* of the lives, or of *two out of three* lives, inſtead of the *ſingle* life in the foregoing rule. This will be more clearly underſtood from the following examples.

EXAMPLE FIRST.

Let the value be required in one preſent payment, and alſo in annual payments of an Aſſurance of 1000 *l.* on two joint lives aged

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by paying the firſt premium *immediately*, and the remaining ones at the *beginning* of every year after.—Under thoſe circumſtances, therefore, it is evident that the proper diviſor will be the value of the life of A for one year *leſs* than the given term added to unity; hence, in the preſent caſe the value of the life of A for 14 years, by Problem 4th and the *Northampton* Table, is 9.387, which being increaſed by unity, and 185.842 *l.* divided by the *ſum*, we have 17.891 *l.* for the *annual* payments, the firſt to be made immediately and the laſt at the beginning of the 15th year.—The ſame is to be underſtood of the examples to the Corollary of this Problem.—And univerſally: The diviſor for determining the *annual* payments muſt be increaſed by unity, whenever it is propoſed that the firſt payment ſhould be made immediately.

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30 for the term of 15 years; that is, let the value be required of 1000*l.* payable if either of two lives aged 30 ſhould fail in 15 years. —The value of the joint lives for 15 years (by Corollary 1ſt, Problem 4th, and the *Northampton* Table) is 8.6896, which ſubtracted from 11.1183 (the value of an Annuity certain for 15 years) leaves 2.4287 the remainder to be *reſerved.*—The value of 1 *l.* to be received at the end of 15 years is .5553. The probability that the life of one or other of two perſons aged 30 ſhall fail in 15 years is $\frac{83600}{189225}$ * (by Table 3d,) and the perpetuity is 25. Theſe numbers multiplied into one another, and 2.4287 added to the product make 8.562, which multiplied by 1000 (the given ſum) and divided by 26 (the perpetuity increaſed by unity) gives 329.308*l.* the required value in *one preſent* payment. And 329.308*l.* divided by 8.6896 (the value of the joint lives for 15 years) gives 37.896 *l.* the ſame value in *annual* payments for 15 years, ſubject to failure ſhould the *joint* lives fail.

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* See note, page 95.

EXAMPLE SECOND.

Let the value be required of an Aſſurance of 1000 *l.* for 15 years on the *longeſt* of two lives aged 30. Or in other words; let the value be required in *one preſent* payment and alſo in *annual* payments of 1000 *l.* to be received on the death of two perſons aged 30, ſhould they both die in 15 years.——The value of the *longeſt* of thoſe two lives for 15 years (by Corollary 1ſt to Problem 4th and the Northampton Table) is 10.904, which ſubtracted from 11.118 (the value of an Annuity certain for 15 years) leaves .214 the remainder to be *reſerved.* The value of 1 *l.* to be received at the end of 15 years is .5553. The probability that the lives of two perſons aged 30 ſhall fail in 15 years is (by Table 3d) $\frac{110}{435}$ multiplied by $\frac{110}{435}$, or $\frac{12100}{189225}$ *; and the perpetuity 25.—Theſe numbers multiplied by one another, and .214 added to the product, make 1.021, which multiplied by 1000 (the given ſum) and divided by 26 (the perpetuity increaſed by unity) gives 39.269 *l.* the anſwer in a *ſingle preſent* payment.—And 39.269 *l.* divided by 10.904 (the value of the longeſt of the two lives for 15 years) gives 3.601 *l.* the anſwer in *annual* payments for 15 years, ſubject to failure ſhould *both* the lives fail.

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* See note, page 95.

EXAMPLE THIRD.

Suppoſe the value is required of 1000 *l.* payable on the extinction of two out of three lives, whoſe ages are 10, 10, and 70, ſhould that happen in 15 years.—By the Corollary to Problem 7th and the *Northampton* Table of Obſervations, the value of an Annuity for 15 years upon two out of the three given lives, is 10.054, which ſubtracted from 11.118 (the value of an Annuity certain for 15 years) leaves 1.064 the remainder to be *reſerved.*—The probability that two at leaſt out of the three given lives ſhall die in 15 years (by Table 3d) is $\frac{427063}{1598508}$ *. This being

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* The probability that two lives at leaſt out of three ſhall die in any number of years is determined in the following manner.—" Find the probability that each *pair* " of lives ſhall fail in the given time; multiply each of " thoſe expreſſions into the probability that the third " life ſhall exiſt ſo long. Find next the probability that " the *three* lives ſhall fail in the given time. Add thoſe " four quantities together, and the *ſum* will give the " probability required."——Thus: Suppoſe the ages of A, B, and C to be 10, 10 and 70 reſpectively; the term

ing multiplied into .5553 (the value of 1 *l.* at the end of 15 years) and alſo into 25 (the perpetuity) as in the two preceding examples will produce 3.713, which added to 1.064, the *reſerved* remainder, becomes equal to 4.777. And 4.777 multiplied into 1000 (the given ſum) and then divided by 26 (the perpetuity increaſed by unity) gives 183.730 *l.* for the value in a *ſingle preſent* payment.

183.73 *l.* divided by 10.054 (the value of an Annuity for 15 years on two out of the three given lives) quotes 18.274 *l.*—the ſame value in *annual* payments for 15 years

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term 15 years, and the probabilities of life to be as they are in the *Northampton* Table of Obſervations, which is the third Table in the Appendix of this work.—By reaſoning as in note, page 95, and note, page 13, the probability

$$\left.\begin{array}{l}\text{That A, B will die}\\ \quad\text{and C live - - -}\\[1ex] \text{That A, C will die}\\ \quad\text{and B live - - -}\\[1ex] \text{That B, C will die}\\ \quad\text{and A live - - -}\\[1ex] \text{That A, B, and C}\\ \quad\text{will all die - - -}\end{array}\right\}\text{will be}\left\{\begin{array}{c}\frac{95}{570}\times\frac{95}{570}\times\frac{17}{123}\\[2ex] \frac{95}{570}\times\frac{106}{123}\times\frac{475}{570}\\[2ex] \frac{95}{570}\times\frac{106}{123}\times\frac{475}{570}\\[2ex] \frac{95}{570}\times\frac{95}{570}\times\frac{106}{123}\end{array}\right\}\text{equal to}\left\{\begin{array}{c}\frac{153425}{39962700}\\[2ex] \frac{4783250}{39962700}\\[2ex] \frac{4783250}{39962700}\\[2ex] \frac{956650}{39962700}\end{array}\right.$$

the ſum of which four expreſſions is $\frac{10676575}{39962700}$, or (dividing the numerator and denominator by 25) $= \frac{427063}{1598508}$.

ſubject to failure ſhould two of the lives fail.

N. B. What is ſaid in note, page 111, reſpecting the method of determining the value of the *annual* payments, is to be applied, under the ſame circumſtances, to the three preceding examples.

PROBLEM X.

To determine the ſum to which any given Annuity, forborn and improved at compound intereſt during the continuance of a given life, will amount at the extinction of that life *.

SOLUTION.

Divide unity by the value of 1 *l.* to be received at the end of as many years as are expreſſed by the *complement*, or twice the expectation, of the given life, and ſubtracting unity from the quotient, *reſerve* the remainder.—Multiply the complement of the given life into the ſquare of the intereſt of 1 *l.* for a year. Divide the *reſerved* remainder

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* See the demonſtration of this Problem in note C. Appendix; where Theorems are alſo given for determining the amount of the ſame Annuity, when it is laid up *half-yearly*, *quarterly*, or *momently*.

der by this product, and from the quotient ſubtract the perpetuity; then will this laſt remainder, multiplied into the given Annuity, produce the value required.

EXAMPLE.

Let the given Annuity be 50 *l.* and the age of the given life 30 years.——By Table 4th the expectation of the life being 28.38, the complement will be 56.76; hence, by Table 1ſt, the value of 1 *l.* to be received at the end of a term equal to this complement is .108.—Unity divided by this number quotes 9.2592, from which unity again being ſubtracted, we have 8.2592 for the remainder to be *reſerved.*——The ſquare of the intereſt of 1 *l.* for a year (or .04 multiplied into itſelf) is .0016; which being multiplied into 56.76 (the complement of the given life) produces .0908; and 8.2592 (the *reſerved* remainder) divided by this product quotes 90.960, from which 25 (the perpetuity) being ſubtracted, and the remainder multiplied by 50 (the given Annuity) we have 3298 *l.* for the value required according to the *Northampton* Table of Obſervations.

The complement of a life at the age of 30, by the *London* Table of Obſervations, is

47.2; and this would have given the amount of the ſame Annuity equal to 2304.2 *l.*—— The complement of the life by Mr. *De Moivre's* hypotheſis being 56, the amount of the Annuity may be found equal to 3209.8 *l.*

PROBLEM XI.

To find the value of an Annuity during a given life, of 1 *l.* payable at the end of the firſt year, 2 *l.* at the end of the ſecond year, 3 *l.* at the end of the third year, and ſo on; the Annuity uniformly increaſing every year till the extinction of the given life.

SOLUTION.

From the value of the given life ſubtract the value of two equal *joint* lives whoſe common age is the age of the given life; and multiply the remainder by the complement of the given life. The product will be the value required.

EXAMPLE.

Let the age of the given life be 40. The value of two equal joint lives aged 40 (by Table in page 74) is 9.71; which ſubtracted from 13.10, the value of the given ſingle life (by Table in page 64) leaves 3.38. And this remainder multiplied by 45.72 (the complement of the given life by Table 4th) gives 154.53 *l.* the required value agreeable to the *Northampton* Table of Obſervations.

COROLLARY FIRST.

If the Annuity is to commence with any larger ſum than 1 *l.* and in the ſame manner to go on increaſing 1 *l.* every year; add to the preceding value, the value of the given life multiplied into the firſt payment leſſened by unity, and the *ſum* will be the anſwer.—Thus; if inſtead of 1 *l.* 2 *l.* 3 *l.* &c. the Annuities payable at the end of the firſt, ſecond, third, &c. years had been 15 *l.* 16 *l.* 17 *l.* &c. 183.47 *l.* (or 13.105 the value of the given life by the Table in page 64 multiplied into 14) muſt have been added to 154.53 *l.* found above, which would have given 338 *l.* for the value in this caſe *.

COROL-

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* It can hardly be neceſſary to obſerve that, if the Annuities at the end of the firſt, ſecond, third, &c. years are conſtantly the *multiples* of any given number, the value in this caſe is equal to the product of ſuch a number multiplied into the value found in the Problem. Thus if the Annuities were 10*l.* 20*l.* 30*l.* &c. or 15*l.* 30*l.*

COROLLARY SECOND.

If the Annuity is not to commence till after an aſſigned time; the value of ſuch an Annuity muſt be found on a life as many years older than the given life as are equal to the aſſigned time.——This value being multiplied into the probability that the life ſhall exiſt ſo long and alſo into 1 *l.* payable at the end of the aſſigned time will give the anſwer.

EXAMPLE.

Let the value be required of an Annuity beginning with 15 *l.* the firſt year, and increaſing at the rate of 1 *l.* every year during the remainder of the life of a perſon now aged 25, after he has attained to the age of 40, and the firſt payment to be made when he has completed his 41ſt year.

The value of ſuch an Annuity on a life of 40 has been found in the preceding Corollary equal to 338*l.* The value of 1 *l.* payable at the

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30*l.* 45*l.* &c. at the end of the firſt, ſecond, third, &c. years, the values in the Example to the Problem would have been equal to (154.53 × 10 =) 1545.3 *l.* and (154.53 × 15 =) 2317.95*l.*

the end of 15 years (by Table 1ſt) is .555, and the probability that a life of 25 ſhall exiſt ſo long (by Table 3d) is $\frac{365}{475}$. Theſe three quantities multiplied into each other give 144.14 *l.* for the value required in *one preſent* payment.

The value in *annual* payments, the firſt to be made immediately and the others at the beginning of each year, is found by dividing the value in *one* payment by the value, added to unity, of the given life for the aſſigned time. Thus; the value of a life aged 25 for 15 years (by the *Northampton* Table and the rule in Problem 4th) is 9.834, which being added to unity, and 144.14 *l.* divided by the *ſum*, we have 13.303 *l.* for the preceding value in *annual* payments, the firſt to be made immediately, and the laſt at the end of the 15th year, or beginning of the 16th year.

COROLLARY THIRD.

If the Annuity, dependent on the given life, is to be continued only for an aſſigned time, beginning with 1 *l.* at the end of the firſt year, and increaſing at the rate of 1 *l.* every year to the end of the term; the value may be found in the following manner.

ner.—“ From the value of the given life
“ for the aſſigned term ſubtract the value
“ for the aſſigned term of two equal *joint*
“ lives, whoſe common age is that of the
“ given life. Multiply the difference by
“ the complement of the given life, and the
“ product will be the anſwer.”

EXAMPLE.

Let the age of the given life be 30, and the aſſigned term 15 years. By Problem 4th and the Table in page 64 the value of a life of 30 for 15 years is 9.7967. By Corollary 1ſt to Problem 4th and the Table in page 74 the value of two equal *joint* lives aged 30 for 15 years is 8.6896. By Table 4th the complement, or twice the expectation, of the given life is 56.76. The difference between the two former values (or 1.107) multiplied into 56.76 gives 62.833*l.* for the value required.

COROLLARY FOURTH.

Suppoſe the Annuity after having increaſed during an aſſigned time is to become *ſtationary*; that is, to continue invariably the ſame for the remainder of the given life after the expiration of the aſſigned time.—In this

caſe the rule for determining the value will be as follows. "Find the value of a life as
" many years older than the given life as are
" included in the aſſigned term. Multiply
" this value into 1 *l.* payable at the end of
" the aſſigned time, and alſo into the proba-
" bility that the given life exiſts ſo long.
" Let this quantity again be multiplied into
" the ſtationary Annuity and *reſerve* the
" product. Find next the value of the in-
" creaſing Annuity by the rule in the fore-
" going Corollary. Let this value be added
" to the *reſerved* product, and the *ſum* will
" be the anſwer."

EXAMPLE.

The value is required of an Annuity on the life of a perſon aged 30, of 1 *l.* payable at the end of the firſt year, 2 *l.* at the end of the ſecond year, 3 *l.* at the end of the third year, and increaſing at this rate every year, till at the end of 15 years it becomes an Annuity of 15 *l.* which is to continue invariably the ſame during the remainder of the given life.——By the Table in page 64 the value of a life aged 45 (or 15 years older than the given life) is 12.2128. The value of 1 *l.* payable at the end of 15 years (by Table 1ſt) is .5553, and the probability that the life will exiſt ſo long (by Table 3d) is $\frac{325}{435}$ *. Thoſe three quantities multiplied into each other are equal to 5.0668; which being multiplied again into 15 (the ſtanding Annuity) gives 76.002 for the product to be *reſerved.*—By the rule in the preceding Corollary the value of the increaſing Annuity for 15 years appears to be 62.833. Therefore 138.835 *l.* is the value required.

COROLLARY FIFTH.

If the Annuity the firſt year is greater than 1 *l,* the value of the life for the given term, till the Annuity either ceaſes or becomes ſtationary, muſt be multiplied into this exceſs above 1 *l.*; and the product, added to the value of the increaſing Annuity for the aſſigned term of 1 *l.* 2 *l.* 3 *l.* &c. will give the anſwer in this caſe. Thus; if inſtead of 1 *l.* 2 *l.* &c. in the Example to Corollary 3d, the Annuity had been 12 *l.* 13 *l.* &c. the value of the life for 15 years found by Problem 4th (or 9.7967) muſt have been multiplied by 11, and then added to 62.833, which would have made the anſwer 170.596 *l.*——In the ſame manner, if the increaſing Annuity in the Example to Corollary 4th had been 12 *l.* 13 *l.* &c. and the *ſtationary* one 26 *l*, 5.0668 in that Example muſt have been multiplied by 26, inſtead of 15, and 131.737, the product, added to 170.596 found above, which would have given 302.333 *l.* for the value in that caſe *.

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* See note, page 95.

COROLLARY SIXTH.

If the Annuity, in the two laſt Corollaries, is not to commence till after an aſſigned time, we muſt proceed exactly in the ſame manner as in the ſecond Corollary. Thus; let the value be required of an Annuity on the life of a perſon now aged 20, and to commence in 10 years, beginning with 1 *l.* the firſt year, and increaſing at the rate of 1 *l.* every year for 15 years; at which period, or when he has attained to the age of 45, it is to continue invariably the ſame for the remainder of his life.—The value of ſuch an Annuity on a life aged 30 appears (by Corollary 4th) to be 138.835. This multiplied into .675 (the value of 1 *l.*

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* If the Annuities 1 *l.* 2 *l.* &c. in the two laſt Corollaries are conſtantly the multiples of any number; ſee note, page 120.

at the end of 10 years by Table 1ſt) and alſo into $\frac{435}{515}$* (the probability by Table 3d, that a life of 20 ſhall exiſt 10 years) gives 79.156*l.* for the value in *one preſent* payment. —And this value divided by 8.466 (the value, increaſed by unity, of the given life for 10 years by Problem 4th) quotes 9.35*l.* for the ſame value in *annual* payments, the firſt to be made immediately and the laſt at the end of the 10th year.

If the Annuity inſtead of becoming *ſtationary* at the end of 15 years after its commencement had entirely *ceaſed,* the value muſt have been found by Corollary 3d, which in the preſent caſe being equal to 62.833, will, when multiplied as above, give 35.824*l.* for the value in *one* payment; and this being divided by 8.466 quotes alſo 4.23*l.* for the ſame value in *annual* payments, the firſt to be made immediately, and the laſt at the end of the 10th year †.

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* See note, page 95.

† Dr. *Price* in his Treatiſe on Reverſionary Payments, page 302, 3d edition, has given another rule for ſolving this Problem. The rule here given agrees with it, but is much ſimpler and eaſier.——See the demonſtration in note D, Appendix.

PROB-

PROBLEM XII.

To determine what Annuity any given ſum will purchaſe during the *joint* lives of two perſons of given ages, and alſo during the life of the *ſurvivor*; on condition, that the Annuity be reduced *one half* at the extinction of the joint lives.

SOLUTION.

Let twice the given ſum be divided by the *ſum* of the two *ſingle* lives, and the quotient will give the Annuity to be paid during the *joint* lives; one *half* of which, therefore, is the Annuity to be paid during the remainder of the ſurviving life *.

EXAMPLE.

A and B whoſe ages reſpectively are 25 and 32, are deſirous of ſinking 1000 *l.* in order to receive an Annuity during their joint lives, and alſo another Annuity of *half* the value during the remainder of the ſurviving life: It is required to determine what Annuities ſhould be granted them under thoſe circumſtances.—By Table in page 64 the value of the life of A is 15.421, and by the ſame Table the value of the life of B is 14.554. 2000 *l.* (or twice the given ſum) being divided by 29.975 (the ſum of the values of the two lives) gives 66.722 *l.* for the Annuity to be granted during the *joint continuance* of the lives; and its half, or 33.361 *l,* is the Annuity to be paid during the life of the ſurvivor.

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* See the demonſtration in note E, Appendix.

COROLLARY.

The value of ſuch an Annuity as is mentioned in the preceding Problem is found, by multiplying *half* the ſum of the ſingle lives into the Annuity to be paid during the *joint* lives. Thus, if A and B, aged 25 and 32, were in poſſeſſion of an Annuity of 40 *l.* to be reduced as above, after the extinction of their joint lives, its preſent value would be equal to 599.48 *l.*; or 14.987 (the half of 29.975) multiplied into 40.

PROBLEM XIII.

To find the value of an Annuity certain for a given term after the extinction of any given life or lives.

SOLUTION.

Subtract the value of the life or lives from the perpetuity, and *reſerve* the remainder. Then ſay; as the perpetuity is to the preſent value of the Annuity certain, ſo is the ſaid *reſerved* remainder to a fourth proportional, which will be the number of years purchaſe required.

EXAMPLE FIRST.

Suppoſe A and his heirs are intitled to an Annuity certain for 14 years to commence at the death of B aged 25. What is the preſent value of A's intereſt in this Annuity?

By Table 5th the value of the life of B is 15.318, which ſubtracted from 25 (the perpetuity) leaves 9.682 for the remainder to be *reſerved*. Therefore as 25 is to 10.563 (the value of an Annuity certain for 14 years by Table 2d) ſo is 9.682 (the *reſerved* remainder) to 4.09, the number of years purchaſe required *.

EXAM-

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* This Solution ſuppoſes the firſt payment of the Annuity to be made at the end of half a year from the extinction of the life. If the firſt payment is not to be made till the end of any longer term, diſcount the value found by this Problem for half a year *leſs* than that term, and the exact anſwer will be obtained.

Thus

EXAMPLE SECOND.

Suppoſe A and his heirs are entitled to an Annuity certain of 40 *l.* for 20 years to commence at the death of the longeſt of two lives aged 30 and 35; what is the value of A's intereſt in this Annuity?——By the rule in page 55, and computing from Tables 5th and 6th, the value of the longeſt of the two lives is 17.825; which ſubtracted from 25 (the perpetuity) leaves 7.175 for the remainder to be *reſerved.*—As, therefore, 25 is to 13.59 (the value of an Annuity certain for 20 years by Table 2d) ſo is 7.175 (the *reſerved* remainder) to 3.9; which multiplied by 40 (the given Annuity) produces 156 *l.* for the value required.

In a ſimilar manner may the value of an Annuity certain after two or three *joint* lives be determined, by ſubtracting the joint lives from the perpetuity, and finding a fourth proportional as above.

Thus, 4.09 diſcounted for a year (that is, multiplied by the value of 1 *l.* to be received at the end of a year) is 3.932, which is the preſent value of an Annuity certain for 14 years, the firſt payment of which is to be made at the end of a year and a half after the failure of a life aged 25.

PROBLEM XIV.

Suppoſing an Annuity be be enjoyed during the life of A, at whoſe deceaſe B and his heirs have the nomination of a ſucceſſor, who is likewiſe to enjoy the Annuity during his life; it is propoſed to find the preſent value of the ſucceeding life.

SOLUTION.

Subtract the value of the life of A from the perpetuity, and *reſerve* the remainder. Then ſay; as the perpetuity is to the preſent value of the life to be nominated at the deceaſe of A, ſo is the ſaid *reſerved* remainder to a fourth proportional, which will be the anſwer.

EXAMPLE.

Let the age of A be 40, and the value of the life to ſucceed him worth 16 years purchaſe, (which nearly correſponds with the value of a life whoſe age is 19 *.)—By Table 5th the value of the life of A is 13.196, which being ſubtracted from 25 (the perpetuity) leaves 11.804 for the remainder to be *reſerved.* Therefore as 25 is to 16 ſo is 11.804 to 7.554, the number of years purchaſe required, according to Mr. *De Moivre*'s hypotheſis.

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* See Table 5th, Appendix, or the Table in page 64.

Had the value of the life of A been taken agreeable to the *Northampton* Table of Obſervations, the value of this ſucceſſion would have been 7.613 *.

REMARK.

This Problem appears to differ in no reſpect from the former, except that in one the ſucceeding Annuity is for a given *life*, in the other for a *term certain.* Conſequently as the Solution in the preceding Problem applied not only to a *ſingle* life but to the *longeſt* of any number of lives, and likewiſe to the *joint* lives, ſo in the preſent Problem the ſame application holds good under ſimilar reſtrictions. This will be better explained by the following example.

EXAMPLE.

An Annuity is enjoyed during the *longeſt* of two lives aged 30 and 35, at whoſe extinction B and his heirs have a right to nominate two other lives, which ſuppoſe to be worth 20 years purchaſe. It is required to determine the preſent value of this nomination. By the rule in page 55 and computing from Tables 5th and 6th the value of the longeſt of two lives aged 30 and 35 is 17.825, which ſubtracted from 25 (the perpetuity) leaves 7.175 for the remainder to be *reſerved.* Therefore, as 25 is to 20 (the ſuppoſed value of the ſucceeding lives) ſo is 7.175 (the *reſerved* remainder) to 5.74, the number of years purchaſe required.

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* A rule for determining the value of any number of ſucceſſive lives may be found in Mr. *Simpſon*'s Select Exerciſes, page 292.

The values of Annuities to one life after one, to one life after two, to two lives after one, to one life after two joint lives, and to two joint lives after one life, may be found by Problems 15th, 18th, 16th, 17th, 19th, reſpectively, in Mr. *Simpſon*'s Select Exerciſes.

COROLLARY.

In order to find the values of the forementioned Annuities for *terms*, all that is neceſſary is to ſubſtitute in Mr. *Simpſon*'s rules, the values of ſingle and joint lives for the *given terms* inſtead of their values for their *whole duration.* Thus for example; ſuppoſe fifteen years of a leaſe to remain unexpired, and that A aged 30 is to enjoy the reverſion of his life in the profits of the leaſe for what may happen to remain of this term after the extinction of the life of B aged alſo 30: It is required to determine the value of A's expectation.

By Mr. *Simpſon*'s 15th Problem the value of the Reverſion of one life after another is obtained, by ſubtracting the two *joint* lives from the life in expectation. Therefore, by Problem 4th, Corollary 1ſt, the value of a ſingle life aged 30 for 15 years according to the *Northampton* Table being 9.7967, and the value of two *joint* lives aged 30 for this term, by the ſame Corollary and Table, being 8.6896, the value of A's expectation will be 1.1071 year's purchaſe.—Had the value of the Reverſion been computed agreeable to Mr. *De Moivre*'s hypotheſis, it would have been found equal to 1.189 years purchaſe.

PROBLEM XV.

To find the value of an Annuity for a term certain, and alſo for what may happen to remain of a given life or lives after the expiration of this term.

SOLUTION.

Find the value of a life or lives as many years older than the given life or lives as are equal to the term for which the Annuity certain is propoſed. Multiply this value by 1 *l.* payable at the end of the given term, and alſo by the probability that the given life or lives will continue ſo long. Add the product to the value of the Annuity certain for the given term, and the *ſum* will be the anſwer.

EXAMPLE FIRST.

Let the value be required of an Annuity certain for 15 years, and alſo for the remainder of a life now aged 30 after the expiration of this term.—By the Table in page 64 the value of a life aged 45 (or 15 years older than the given life) is 12.2128. The value of 1 *l.* payable at the end of 15 years is (by Table 1ſt) .5553; and the probability that the life will exiſt ſo long (by Table 3d) is $\frac{325}{435}$ *. Theſe three numbers multiplied into each other produce 5.0668, which being added to 11.1183 (the value of an Annuity certain for 15 years by Table 2d) becomes equal to 16.1851, the number of years purchaſe required.

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* See note, page 95.

EXAMPLE SECOND.

Suppoſe the value is required of an Annuity certain for 20 years, and alſo for the remainder of the *longeſt* of two lives now aged 10 and 70 after this term.——By the rule in page 55 and calculating according to the values of ſingle and joint lives in pag. 64 and 78, the value of the longeſt of two lives aged 30 and 90 (or 20 years older than the given lives) may be found equal to 14.8787. The value of 1 *l.* payable at the end of 20 years (by Table 1ſt) is .4564, and the probability (by Table 3d) that two lives aged 10 and 70 ſhall not *both* fail in 20 years is $\frac{54045}{70110}$ *. Theſe three numbers multiplied into each other produce 5.234, which being added to 13.590 (the value of an Annuity certain, by Table 2d, for 20 years) is equal to 18.824, the value required.

PROBLEM XVI.

B, who is of a given age will if he lives till the deceaſe of A, whoſe age is alſo given, become poſſeſſed of an eſtate of a given value; to find the worth of his expectation in preſent money.

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* See note, page 95.

SOLU-

SOLUTION.

Find the value of an Annuity on two equal *joint* lives whoſe common age is equal to the age of the oldeſt of the two propoſed lives, which value ſubtract from the perpetuity, and take *half* the remainder; then ſay, as the expectation of duration of the younger of the two lives is to that of the older, ſo is the ſaid half remainder to a fourth proportional; which will be the number of years purchaſe required when the life of B in expectation is the older of the two: But if B be the younger, then add the value ſo found to that of the joint lives A and B, and let the ſum be ſubtracted from the perpetuity, and you will alſo have the anſwer in this caſe.

The value of the Reverſion in *annual* payments is determined by dividing the *ſingle preſent* payment by the value of the *joint* lives.

EXAMPLE FIRST.

Suppoſe the age of A to be 20 and that of B 30 years; and the given legacy 10 *l. per ann.* Then the value of two equal joint lives aged 30 (by Table 6th) being 11.182, and the perpetuity 25; the difference here will be 13.818, the *half* of which is 6.909. Therefore as 33 *, the expectation of A, is to 28, the expectation of B, ſo is 6.909 to 5.862 years purchaſe, which being multiplied by 10, the given Annuity, we have 58.62 for the required value of B's expectation in *one preſent* payment.

This is the anſwer by Mr. *De Moivre's* hypotheſis. But if an anſwer is required by the *Northampton* Table of Obſervations, the value of the two equal joint lives will (by the Table, page 74) be 11.450. The expectation of the youngeſt life will (by Table

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* The values of moſt of the following examples in this chapter have been computed from Mr. *De Moivre's* hypotheſis; but it will be found as I go along that the Rules may be applied to any Table of Obſervations; always remembering that the *complement* of a life, according to any Table, is double its *expectation* of duration.

It ſhould be further carefully remembered that the Rules for ſolving this and all the following Problems give the *exact* values according to Mr. *De Moivre's* hypotheſis (except when the approximations in Problems 5th and 6th are uſed for finding the values of three *joint* lives, or three *ſingle* lives †;) but that they are only *approximations* when the complements and values of lives are taken (as it is beſt they ſhould) agreeable to a given Table of Obſervations.——Theſe approximations, however, are ſo near when the Survivorſhip extends to the whole duration of the lives, that there is little reaſon to deſire any greater accuracy.

† See Scholium to Problem 6th.

4th, Appendix) be 33.19, and of the olde 28.38.—And theſe data will bring out th anſwer 57.93 *l.*

EXAMPLE SECOND.

Let the age of A be 30, that of B 20 years; and the reſt as in the preceding example. Here the value of the joint lives A and *B* (by Table 6th) is 11.711, which being added to 5.862, found above, the ſum is 17.573; and this ſubtracted from 25, the perpetuity, and multipled into 10, gives 74.27 *l.* for the value in *one preſent* payment. —And 74.27 being divided by 11.711 (the value of the joint lives A and B) quotes 6.34 *l.* for the ſame value in *annual* payments, the firſt to be made at the *end* of the year.

PROBLEM XVII.

To find the value of a given eſtate at the death of B, provided that ſhould happen *after* the death of A.

SOLUTION.

Find the value of an Annuity upon the *longeſt* of two equal lives whoſe common age is that of the older of the two lives A nd B, which value ſubtract from the perpetuity and take *half* the remainder. Then it vill be, as the expectation of duration of the ounger of the lives A and B is to that of he older, ſo is the ſaid half-remainder to he number of years purchaſe required when B is the older of the two. But if B be the ounger, then to the number of years purchaſe thus found add the value of an Annuity on the longeſt of the lives A and B, nd ſubtract the ſum from the perpetuity for the anſwer in this caſe.

The value of the Reverſion in *annual* payments till the *claim is determined* is found by dividing the *ſingle preſent* payment by the value of the two *joint* lives.—And the ſame value in *annual* payments till the claim *becomes due* is found by dividing the ſingle payment by the life of B.

EXAMPLE.

Let the age of A be 30, and that of B 60 years. Let the given eſtate alſo be 40 *l.* *per annum.*——By the rule in page 55 and calculating from Tables 5th and 6th, Appendix, the value of the *longeſt* of two lives aged 60 each will be found equal to 11.683, which taken from 25, the perpetuity, leaves

13.317 for the remainder. Therefore it will be as 28, the expectation of A, is to 13, the expectation of B, ſo is 6.658, the half-remainder, to 3.091, the number of years purchaſe required, which being multiplied by 40 gives 123.64*l.* for the value in *one preſent* payment.

But if A had been 60 and B 30, the value required would have been 5.987 multiplied into 40, or 239.48*l.*; becauſe the value of the longeſt of the lives A and B (by the rule in page 55 and Tables 5th and 6th, Appendix) is 15.922 years purchaſe.

Such is the ſolution by Mr. *De Moivre's* hypotheſis.—Had the ſolution been required by the *Northampton* Table, it would have been neceſſary to compute (by Problem 3d) the value of two joint lives aged 30 and 60 from their expectations by the ſame Table, or from 28.38 and 13.13*; which differing but little from the expectations by the hypotheſis, the reſult would have been very nearly the ſame.—But had the ages of A and B been ſuppoſed 10 and 70, the value by the hypotheſis (ſuppoſing A the oldeſt) would have been 5.92 years purchaſe or

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* See Table 4th, Appendix.

236.8 *l.*—By the *Northampton* Table * 5.45 years purchaſe or 218 *l.*——Had both the ages been 10 the value would have been by the hypotheſis 2.29 years purchaſe.—By the *Northampton* Table 2.14 years purchaſe.—Had the age of A been 75 and B 15, the value by the hypotheſis would have been 6.87. By the *Northampton* Table 6.43 years purchaſe.

It appears from hence that, though the *Northampton* Table correſponds more nearly than any other to the hypotheſis, yet the difference between the values given by it and by the hypotheſis in the firſt and laſt ſtages of life is too great to be neglected.—And what is thus true of this Table and alſo of Dr. *Halley*'s Table in the firſt and laſt ſtages of life, is true of the *London* Table through *every* part of life.

REMARK.

If inſtead of an *Annuity* the value of an equivalent *ſum* had been required, the reſpective anſwers in the two preceding Problems ſhould have been divided by 1 *l.* increaſed by its intereſt for a year, agreeable to the correction mentioned by Dr. *Price* in his Treatiſe on Reverſionary Payments, Scholium to Queſtion 6th.

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* See Table 4th, Appendix, and the Tables in pag. 64, 74, and 78.

The ſame is likewiſe to be obſerved in all the following ſolutions where either of thoſe Problems are concerned.

The Solutions of the 16th and 17th Problems include all the caſes of Survivorſhip between two lives for their *whole duration*. But an expectation dependent on a Survivorſhip is often reſtricted to a *term of years* leſs than the whole duration of the lives.—I am unwilling to repeat too much of what has been written by others; and, therefore, I ſhall refer for the Rules, by which the values in ſuch caſes may be calculated, to Queſtions 15th and 16th in Dr. *Price*'s Treatiſe on Reverſionary Payments.

PROBLEM XVIII.

B if he lives till the deceaſe of A is to receive a given ſum in caſe C is then extinct. —To determine the value of his expectation in preſent money.

SOLUTION.

CASE 1ſt. *If the life of B is the oldeſt of the three:* From the value of an Annuity on the life of B take the value of the two joint lives A and B; multiply the remainder by the given ſum, and divide the product by twice C's expectation; the quotient will be the value ſought.

CASE 2d. *If the life A is the oldeſt of the three:* From the value of an Annuity for as many years of B's life as are expreſſed by the double of A's expectation of duration (found by Problem 4th) ſubtract the value of the two joint lives A and B; and multiply the remainder by the given ſum, then divide the product by twice the expectation of C's duration as in the preceding caſe.

CASE 3d. *If the life C be the oldeſt of the three:* Then find the value of the life B, if older than A; otherwiſe find (by Problem 4th) the value of as many years thereof as are expreſſed by the double of A's expectation of duration; and from the value thus found let the value of the joint lives C and B be ſubtracted: Multiply the remainder by the given ſum, then the product divided by twice A's expectation of duration will be the anſwer.

In either of thoſe three caſes if the preſent value in *one* payment be divided by the joint lives A and B, the quotient will be the ſame value in *annual* payments *, the firſt to be made at the end of the year and the others to be continued till the claim is determined; or (which is the ſame thing in this inſtance) till the money becomes payable, provided the claim is determined in favour of the purchaſer.

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* See the method by which the values in *annual* payments, in all the following Problems, have been deduced, in note K, Appendix.

EXAMPLE FIRST.

Let the age of B be 60, that of A 20, and that of C 30 years; and let the propoſed legacy be 1000 *l.*—This example, it is evident, belongs to Caſe 1ſt. Hence 7.967, the value of the joint lives A and B, by Table 6th, being ſubtracted from 9.017, the value of the ſingle life B by Table 5th, leaves 1.05 for the remainder; which being multiplied into 1000, the given ſum, and divided by 56, twice the expectation of C by the hypotheſis, quotes 18.75 *l.* for the value required in *one preſent* payment.—In order to find this value in *annual* payments till the claim is determined or becomes payable, 18.75 *l.* muſt be divided by 7.967, the value of the joint lives A and B, and the quotient, or 2.353 *l*, will be the anſwer.

Had the ages of A, B, and C reſpectively been 10, 70, and 50, the value of B's expectation in *preſent* money would have been by the *Northampton* Table of Obſervations equal to 9.752 *l*, and by Mr. *De Moivre's* hypotheſis equal to 11.194 *l*.

EXAMPLE SECOND.

Let the age of B be 20, that of A 60, and that of C 30 years; and let the propoſed legacy be 1000 *l*.——According to the ſolution of Caſe 2d the firſt thing to be determined is the expectation of A, which by the hypotheſis appears to be 13; its double, therefore, is 26, and the value of an Annuity on the life of B for 26 years by Problem 4th, and computing agreeable to the hypotheſis, is equal to 13.235, from which 7.967, the value of the joint lives A and B by Table 6th, being ſubtracted, the remainder is 5.268; which multiplied into 1000, the given ſum, and then divided by 56, twice the expectation of C, quotes 94.071 *l*. for the value required.—And 94.071 divided by 7.967, as above, gives 11.807 *l*. for the ſame value in *annual* payments, the firſt to be made at the end of the year.

EXAMPLE THIRD.

Suppoſe the age of B to be 30 years, that of A 20 and that of C 60. The legacy, as in the other examples, 1000 *l*.—According to Caſe 3d the value of the life of B (by Table 5th) is 14.684, from which 7.779 the value of the joint lives C and B (by Table 6th) being ſubtracted, and 6.905, the remainder, multiplied into 1000, the given ſum, and then divided by 66, twice the expectation of A by the hypotheſis, quotes 104.62 *l.* for the value ſought. And 104.62 *l.* divided by 11.7, the value of the joint lives A and B by Table 6th, quotes 8.942 *l.* for the ſame value in *annual* payments, the firſt to be made at the *end* of the year.

PROBLEM XIX.

To find the value of a given ſum payable if A ſhould be the *firſt* that fails of the three lives A, B, and C.

SOLUTION.

Find by Problem 18th the value of the given ſum payable if C (the elder of B and C) ſhould be living at the deceaſe of A in caſe B is then dead.—Find next by Problem 16th the value of the given ſum payable if C is living at the deceaſe of A whether B is then living or not.—Subtract the former value from the latter, and the remainder will be the anſwer.

If the value is required in *annual* payments to be continued till the claim is either determined or becomes due, the preceding anſwer muſt be divided by an Annuity on the joint continuance of the three lives.

EXAMPLE.

Let the age of A be 20 years, that of B 30, and that of C 60. The ſum 1000 *l.*— Then by Example 1ſt, Problem 18th, the firſt value appears to be 18.75; and by Problem 16th the ſecond value is 141.26. Their difference, therefore, or 122.51 *l.* will be the value required.—And 122.51 being divided by 6.88, the value of the three joint lives by Problem 5th, the quotient 17.8 *l.* is the ſame value in *annual* payments, the firſt to be made at the *end* of the year.

PROBLEM XX.

To find the value of a given ſum payable if A ſhould be the *ſecond* that fails of the three lives A, B, and C.

SOLUTION.

Find by Problem 18th the value of the given ſum payable if B ſhould live to the deceaſe of A, and C is then extinct. Find alſo by the ſame Problem the value of the given ſum payable if C ſhould live to the deceaſe of A, and B is then extinct. The ſum of theſe two values will be the anſwer in *one* preſent payment.

To find the value in *annual* payments till the claim is either determined or becomes due; ſubtract the value of an Annuity on the three joint lives from the *ſum* of the values of the two joint lives A and C, and the two joint lives A and B. Divide the *ſingle* payment by this remainder, and the quotient will be the anſwer.

EXAMPLE.

Let the age of A be 20 years, that of B 30, and that of C 60.—The ſum 1000 *l.*—By Example 3d, Problem 18th, the firſt value appears to be 104.62; and by Example 1ſt the latter value appears to be 18.75. Their ſum, therefore, or 123.37 *l.* is the anſwer in *one preſent* payment.

The value of the joint lives A and B (by Table 6th) is 11.711, of the joint lives A and C (by the ſame Table) 7.967, and of the three joint lives A, B, and C (by Problem 5th) 6.882. This laſt value being ſubtracted from the ſum of the two former values (or 19.678) leaves 12.796; and

123.37 *l.* divided by this remainder quotes 9.641 *l.* for the value in *annual* payments, the firſt to be made at the *end* of the year.

PROBLEM XXI.

To find the value of a given ſum payable at the death of A, if his life ſhould be the *laſt* that fails of the three lives A, B, and C.

SOLUTION.

Find by Problem 18th the value of the given ſum payable if C ſhould live till the deceaſe of A in caſe B ſhould be then dead (C being ſuppoſed the elder of B and C.) Find next by Problem 17th the value of the given ſum payable if A ſhould die *after* B. Subtract the former value from the latter, and the remainder will be the anſwer in *one preſent* payment.

The value in *annual* payments till the claim *is determined* is found by the rule in Problem 20th.—And the ſame value till the claim *becomes due* (provided it ſhould be determined in favour of the purchaſer) is found by dividing the ſingle payment by an Annuity on the life of A.

EXAMPLE.

Let the age of A be 60 years, that of B 20, and that of C 30. The ſum 1000 *l.*— By Caſe 2d, Problem 18th, the firſt value will be found equal to 75.288, and by Problem 17th the latter value equal to 100.877; their difference, therefore, or 25.589 *l.* is the value required in *one preſent* payment.

By proceeding according to the rule in Problem 20th the difference between the three joint lives, (found by Problem 5th) and the *ſum* of the values of the two joint lives A and C, and the two joint lives A and B (by Table 6th) will be found equal to 8.864. And 25.589 *l.* divided by this difference quotes 2.886 *l.* for the value in *annual* payments, to be made at the *end* of every year till the claim *is determined.*

The value of the ſingle life A (by Table 5th) is 9.02, and 25.589 *l.* divided by this Annuity gives 2.836 *l.* for the *annual* payments to be made as above till the claim *becomes due.*

Theſe, it may be obſerved, are the values agreeable to Mr. *De Moivre's* hypotheſis. Had the reſpective ages of A, B, and C been 10, 10, and 70, the value of the Reverſion, by the ſame hypotheſis, in *one preſent* payment, would have been equal to 82.643*l.* and by the *Northampton* Table of Obſervations equal to 79.416 *l.* *

PROBLEM XXII.

To find the value of a given ſum payable at the extinction of the lives A and B, ſhould they be the *firſt* that will fail of the three lives A, B, and C.

SOLUTION.

Find by Problem 18th the expectation of C on the contingency of A's ſurviving B, and alſo his expectation on the contingency of B's ſurviving A. The *ſum* will be the value required in *one preſent* payment.

To obtain the ſame value in *annual* payments till the claim is either *determined* or *becomes due*; find the value of the two joint lives B and C, and alſo the value of the two joint lives A and C, from the *ſum* of which ſubtract the value of the three joint lives; then divide the ſingle payment by the remainder, and the quotient will be the anſwer.

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* See the latter part of the Example to Problem 17th.

EXAM-

EXAMPLE.

Let the age of A be 20 years, that of B 60, and that of C 30. The ſum 1000 *l.*—Here the firſt part of C's expectation appears from Example 3d, Problem 18th, (by ſubſtituting C for B and *vice verſâ)* to be 104.62 *l.*; and the latter part of his expectation is found by Caſe 2d (ſubſtituting C for B, A for C, and B for A) to be 75.288 *l.* Therefore 179.908 *l.* is the anſwer in *one preſent* payment.

Again.—The value of the joint lives B and C (by Table 6th) is 7.779, and of the joint lives A and C (by the ſame Table) 11.711; from the *ſum* of which 6.882 (the value of the three joint lives by Problem 5th) being ſubtracted, we have 12.6 for the remainder: And 179.908 *l.* being divided by 12.6 quotes 14.278 *l.* for the value of the above expectation in *annual* payments to be made at the end of every year till the claim is determined or becomes due.

PROBLEM XXIII.

To find the preſent worth of an eſtate of a given yearly value, to be entered upon at the *deceaſe** of A and B, ſhould their lives be the *laſt* that ſhall fail of the three lives A, B, and C.

SOLUTION.

CASE 1ſt. *If either A or B are older than C.* Find by Problem 6th the value of an Annuity on the longeſt of three equal lives whoſe common age is that of the elder of the two lives A and B. Subtract this value from the perpetuity; divide the remainder by 3 and *reſerve* the quotient.—Find next the value of 1 *l.* payable at the end of as many years as are expreſſed by the complement, or twice the expectation, of the youngeſt of A and B; which being multiplied into half the perpetuity, let the product be ſubtracted from the *reſerved* quotient. Then it will be, as the products of the *complements* of C and the youngeſt of the two lives A and B, to the ſquare of the *complement* of the oldeſt of them, ſo is the preceding remainder to a fourth proportional, which muſt be *reſerved.*

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* Care ſhould be taken to diſtinguiſh between this and the queſtion "for finding the value of an eſtate, or "equivalent ſum, to be received *as ſoon as it is determined* whether A and B ſhall be the *laſt* that will fail "of the three lives A, B, and C." This is the ſame with the 17th Problem and differs widely from the 23d; for it is evident that in the one the expectant is to become poſſeſt of the Reverſion on the extinction of the *joint* lives, but that in the other he cannot come into poſſeſſion till the extinction of the *longeſt* of the lives.

Find next by Problem 4th the value of an Annuity on the life of the youngeſt of A and B for a term equal to the *difference* of the *complements* of thoſe two lives, which ſubtract from the perpetuity and take half the remainder. Then it will be, as the *complement* of C is to the *complement* of the eldeſt of A and B, ſo is the ſaid half remainder to a fourth proportional; which being multiplied into the value of 1 *l.* payable at the end of a term equal to the *complement* of the eldeſt of A and B, and added to the *reſerved* proportional, the *ſum* will be the number of years purchaſe required.

CASE 2d. *If C is the oldeſt of the three.* Find by the preceding rule the value of the eſtate to be entered upon at the deceaſe of A and C, if *B* ſhould be the firſt that fails. Find alſo the ſame value at the deceaſe of B and C if *A* ſhould be the firſt that fails. Add theſe two values to the value of an Annuity on the *longeſt* of the three lives A, B, and C (found by Problem 6th;) then ſubtract the *ſum* from the perpetuity, and the remainder will be the number of years purchaſe required.

If in either of thoſe two caſes, the number of years purchaſe be multiplied by any given *ſum*, and divided by the perpetuity increaſed by unity, the quotient will be the value of the Reverſion of this ſum *.

The above value in *annual* payments till the claim is *determined* may be obtained, by dividing the value in *one* payment by the three joint lives (found by Problem 5th.)

In order to obtain the value in *annual* payments till the money *becomes due*; that is, till the claim is determined, and afterwards till it becomes payable provided it ſhould be determined in favour of the purchaſer, divide the value in *one* payment by the value of an Annuity on the three *joint* lives added to the *difference* between the value of an Annuity on the life of C and on the *longeſt* of the three lives A, B, and C.

EXAMPLE FIRST.

Let the age of A be 20 years, that of B 50, and that of C 30. The eſtate, 40 *l.* *per annum*, which is equivalent to the ſingle ſum of 1000 *l.*—This example, it is plain, belongs to Caſe 1ſt.——The value of the longeſt of three lives aged 60 is 12.987, which ſubtracted from 25 (the perpetuity) and the remainder divided by 3 becomes equal to 4.004. The value of 1 *l.* at the end of 66 years, or a term equal to the complement of A, is (by Table 1ſt) .075, which multiplied into half the perpetuity, or 12.5, is equal to .9375; and this ſubtracted from 4.004 leaves 3.0665. Therefore as 56 (the complement of C) multiplied into 66 (the complement of A.) or 3696, is to the ſquare of 26 (the complement of B,) or 676, ſo is 3.0665 to .5607, the fourth proportional to be *reſerved.*——Again. The value of an Annuity on the life of A for ($\overline{66-26}=$) 40 years is 15.151, which ſubtracted from 25, and the remainder divided by 2, becomes equal to 4.924. Hence as 56 is to 26 ſo is 4.924 to 2.29; and this proportional multiplied by .3607, the value of 1 *l.* (by Table 1ſt) at the end of 26 years, is reduced to .825. Conſequently 1.3857 is the number of years purchaſe required; which being multiplied by 1000 and divided by 26 quotes 53.296 *l.* for the value of the given *equivalent ſum* in *one preſent* payment.

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* See the demonſtration of this Problem in note F, Appendix.

The value of the three *joint* lives is 6.882, and 53.296 *l.* divided by this quantity quotes 7.744 *l.* for the value of the above ſum in *annual* payments till the claim is *determined*.

The value of the three *joint* lives being 6.882, the value of the *longeſt* of the three lives (by Problem 6th) 19.165, and the value of the life of C (by Table 5th) 14.684; the *difference* between the latter and the *ſum* of the two former values will be 11.363, and 53.296 *l.* divided by this difference quotes 4.69 for the value in *annual* payments to be continued till the claim *becomes due* *.

Such are the values by Mr. *De Moivre's* hypotheſis. Suppoſe now the value of the preceding Reverſion in *one preſent* payment is required agreeable to the *Northampton* Table of Obſervations. By Scholium to Problem 6th it appears that the *exact* value of the longeſt of three lives aged 60 is 13.1013. By Table 4th, Appendix, the complements of the lives of A, B, and C are 66.38, 26.26, and 56.76 reſpectively. Therefore following the ſame ſteps as in the foregoing Example the *reſerved* proportional will be found equal to .5565.——The value of an Annuity on the life of A for ($\overline{66.38-26.26}=$) 40 years, by Problem 4th and the Table in page 64, is 15.2; conſequently the number to be added to the *reſerved* proportional is .805, which makes the value of the *eſtate* to be 1.3615 years purchaſe; hence the value of the given ſum is 52.365 *l.*

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* If inſtead of making the *annual* payments at the *end* of every year, they are to be made at the *beginning* of each year, *unity* muſt be added to the diviſors for determining thoſe values;—that is, in the preſent caſe 53.296 *l.* muſt be divided either by 7.882 or 12.363.

EXAMPLE SECOND.

Let the age of A be 20 years, that of B 30, and that of C 60.—The ſum 1000 *l.* This Example belongs to Caſe 2d; therefore the firſt value by ſubſtituting C for B and *vice verſâ*, appears by the preceding Example to be 1.3857; and by following the directions laid down in Caſe 1ſt (after ſubſtituting C for A and *vice verſâ*) the ſecond value will be found equal to 1.2898. Their *ſum* added to 19.165, the value of the longeſt of the three lives, and then ſubtracted from 25, the perpetuity, leaves 3.1595 for the number of years purchaſe required; and conſequently the value of the given ſum is 121.52 *l.*

The value of the three *joint* lives being 6.882, the above value in *annual* payments, to be made at the *end* of every year till the claim *is determined,* will be 17.657*l.*—And the *difference* between the life of C (by Table 5th) and the *ſum* of the values of the three *joint* lives and the *longeſt* of the three lives being 17.03, the *annual* payments, to be made at the end of every year till the claim *becomes due,* will be 7.135 *l.*

PROBLEM XXIV.

To find the value of a given ſum payable at the deceaſe of A provided *both* or *either* of the two lives B and C are then living; that is, if the life of A ſhould be the *firſt* or *ſecond* that ſhall fail of the three lives A, B, and C.

SOLUTION.

Find by Problem 21ſt the value of the given ſum payable if A ſhould be the *laſt* that dies of the three lives: Subtract this from the value of the Reverſion of the given ſum after the deceaſe of A (found by Problem 8th) and the remainder will be the anſwer in *one preſent* payment.

The ſame value in *annual* payments, till the claim is either *determined* or *becomes due*, will be obtained by dividing the value in one payment by an Annuity on the three joint lives ſubtracted from the ſum of the values of an Annuity on the joint lives of A and C and an Annuity on the joint lives of A and B.

EXAMPLE.

Suppoſe the age of A to be 20 years, that of B 30, and that of C 60. The ſum 1000*l.*—The firſt value by Problem 21ſt may be found equal to 106.096 *l.* and the ſecond value (by Problem 8th) equal to 350.346 *l.* therefore the value required in *one* payment, by the hypotheſis, is 244.25 *l.*

The value of the three joint lives, by the ſame hypotheſis and Problem 5th, is 6.882, which ſubtracted from 19.678, the *ſum* of the values of the joint lives A and C, and the joint lives A and B (by Table 6th) leaves 12,796. Hence 244.25 divided by this remainder, quotes 19.088*l.* for the value in *annual* payments to be made at the end of every year till the claim is either determined or becomes due.

PROBLEM XXV.

To find the value of a given ſum payable at the deceaſe of A if his life ſhould *not* be the firſt that ſhall fail of the three lives A, B, and C; that is, if it ſhould be the *ſecond* or *third* that fails of thoſe lives.

SOLUTION.

Find by Problem 19th the value of the given ſum if A ſhould be the *firſt* that dies: Subtract this value from the value of the Reverſion of the given ſum after A's life (found by Problem 8th,) and the remainder will be the anſwer in *one preſent* payment.

The value in *annual* payments till the claim *is determined*, is found by the rule in Problem 20th. And the value in thoſe payments till the claim *becomes due* is found by dividing the ſingle payment by the value of an Annuity on the life of A.

EXAMPLE.

Let the age of A be 20 years, that of B 30, and that of C 60. The ſum 1000 *l.*—The value on the firſt contingency (by Problem 19th) will be found equal to 122,51 *l.* and the value of the Reverſion after A's life (by Problem 8th) is 350.346 *l.*; conſequently 227.836 *l.* will be the value required, by the hypotheſis, in *one* payment.

227.836 *l.* divided by 12.796 (found by the rule in Problem 20th) quotes 17.805 *l.* for the value in *annual* payments to be made at the end of every year till the claim is *determined*. And 227.836 *l.* divided by 15.89 (the value of an Annuity on the life of A by Table 5th) gives 14.33 *l.* for the value in thoſe payments, to be made in the ſame manner till the claim *becomes due*.

PROBLEM XXVI.

To find the value of a given ſum payable at the deceaſe of A, if his life ſhould *not* be the ſecond that fails; that is, if it ſhould be the *firſt* or *laſt* that ſhall fail of the three lives A, B, and C.

SOLUTION.

Find by Problem 20th the value of the given ſum provided A ſhould be the *ſecond* that dies: Subtract this value from the value of the Reverſion after A's life (found by Problem 8th) and the remainder will be the anſwer in *one preſent* payment.

The value in *annual* payments till the claim is *determined* is found as in Problem 20th.—The value in *annual* payments till the claim *becomes due* (provided it ſhould be determined in favour of the purchaſer) is the value in *one* payment divided by the value of the ſingle life A.

EXAMPLE.

Suppoſe the age of A 20, that of B 30, and that of C 60 years. The ſum 1000 *l.* The firſt value according to Mr. *De Moivre*'s hypotheſis appears to be 123.34 *l.* and the ſecond value 350.346 *l.* Their difference, therefore, or 227.006 *l.* is the value required in *one preſent* payment;—which being divided by 12.796 (found by the rule in Problem 20th) quotes 17.74 *l.* for the value in annual payments to be made at the end of every year till the claim is determined: and being divided by 15.89 (the value of an Annuity on the life of A by Table 5th) quotes 14.286 *l.* for the value in thoſe payments to be made in the ſame manner till the claim *becomes due*, provided it ſhould be determined in favour of the purchaſer.

PROBLEM XXVII.

To find the value of an Annuity on the life of B after A, on the particular condition that A's life when it fails, ſhall fail *before* the life of C.

SOLUTION.

Case 1ſt. *When B is oldeſt; or when B is older than C, and C is older than A.*—From the value of an Annuity on the two joint lives B and C, ſubtract the value of an Annuity on the three joint lives A, B, and C; and *reſerve* the remainder.

Find by the rule in page 55 the value of an Annuity on the longeſt of the two lives A and C, and by Problem 6th, or its Scholium, the value of an Annuity on the longeſt of the three lives A, B, and C; then will *half* the difference between theſe two laſt values added to the *reſerved* remainder be the number of years purchaſe required.

Case 2d. *When B is younger than C, and C older than A.* Find, as in the preceding Caſe, the difference between the values of the two joint lives B and C, and the three joint lives A, B, and C, and let it be *reſerved.*—Find next the value of an Annuity on the longeſt of two equal lives whoſe common age is that of C, and ſubtract it from the value of an Annuity on the longeſt of three lives, the common age of two of whom is that of C, and the age of the third that of B; multiply this remainder by the expectation of C, divide the product by twice the expectation of A, and the quotient added to the remainder, *reſerved* above, will be the number of years purchaſe required.

CASE 3d. *When B is younger than C, and C younger than A.* Find the value as in Caſe 2d, upon the ſuppoſition that C ſhould die before A. Add this value to that of an Annuity on the three joint lives; let the ſum be ſubtracted from the value of an Annuity on the ſingle life B, and the remainder will give the number of years purchaſe required.

CASE 4th. *When B is older than C, and A older than B.* This is anſwered in the ſame manner with Caſe 3d, by firſt finding the value of an Annuity by Caſe 2d on the life of B after C on condition that C dies before A, and then ſubtracting this value, added to the value of the three joint lives, from the ſingle life B *.

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* See the demonſtration of this Problem in note G, Appendix.

EXAMPLE FIRST.

Let the age of B be 60 years, that of C 30, and that of A 20.——The value of the joint lives of B and C (by Table 6th) is 7.779, and the value of the three joint lives (by Problem 5th and the hypotheſis) is 6.882; therefore the remainder to be *reſerved* is .897.

The value of the longeſt of the three lives (by Problem 6th and the hypotheſis) is 19.165, and that of the longeſt of the two lives A and C (by the rule in page 55) is 18.864; conſequently half their difference is .15, which added to .897 gives 1.047 for the number of years purchaſe required *.

The

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* The anſwers in this and the following Examples are a little *above* the truth, owing to the value of the three joint lives, and that of the longeſt of the three lives being given by Prob. 5th and 6th, the one too high and the other too low. Theſe deviations ariſe from the rules in thoſe Problems being only *approximations* to the real values.—But they may always be uſed with ſafety, provided we approximate to the values of ſingle and joint lives (by Prob. 2d and 3d) from a *given rate of mortality* when the ages exceed 60; Mr. *De Moivre*'s hypotheſis, (as I have already obſerved in the Scholium to the ſixth Problem) producing errors too conſiderable after this period.—Having regard to this precaution, we ſhall ſeldom or ever deviate from the truth more than $\frac{1}{10}$th of a year's purchaſe.

—I ſhould

The value of this Reverſion by the *Northampton* Table, and approximating to the values of the ſingle and joint lives by the rules in Prob. 2d and 3d, is .88.

EXAMPLE SECOND.

Let the age of B be 20 years, that of C 60, and that of A 30.—The value of the joint lives B and C (by Table 6th) being 7.967, and the value of the three joint lives (by Problem 5th) being 6.882, the difference

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—I ſhould not have taken notice of theſe circumſtances, had it not been that, in the preſent Example, the value of the life of B after A, though reſtrained to the contingency of C's ſurviving A, appears to be almoſt equal to the value of the life of B after A without any ſuch reſtriction; from whence the reader might be led to conclude that the rule is wrong.—Now it is evident that when the ages of the lives are reſpectively as in this Caſe, there cannot be much difference between them.—If the values, however, of the three joint lives and the longeſt of the three lives be computed in exact agreement to the hypotheſis by Problem 1ſt, Corollary 5th, and Problem 2d, Corollary 2d, in *Simpſon's* Doctrine of Annuities, they will be found equal to 6.949 and 19.084, which will reduce the value of the Reverſion to .94, or nearly $\frac{1}{10}$th of a year's purchaſe below the value given in the Example.—This reduced value may be proved to be accurately true from other principles.—See *Simpſon's* Select Exerciſes, Problem 35th, where a different ſolution is given of this Problem, but which agrees in the reſult exactly with the above.

to be reſerved is 1.085.—The value of an Annuity on the longeſt of the three lives B, C, and C (by Problem 6th and *De Moivre's* hypotheſis) is 17.445, and on the longeſt of the two lives, whoſe common age is that of C, (by the rule in page 55 and the ſame hypotheſis) 11.683: The latter value ſubtracted from the former leaves 5.762; which multiplied by 13 (the expectation of C) and then divided by 56 (twice the expectation of A) will be equal to 1.337; and this quantity added to 1.085, *reſerved* above, gives 2.422 for the anſwer in this caſe.

The value of this Reverſion by the *Northampton* Table and the approximations in Problems 2d and 3d, is 2.429.

EXAMPLE THIRD.

Let the age of B be 20, that of C 30, and that of A 60 years. The value of the Reverſion (by Caſe 2d) upon the ſuppoſition that C ſhould die before A is 2.422. The value of the three joint lives is 6.882; and both theſe added together and then ſubtracted from 15.891 (the value of the life of B by Table 5th) will leave 6.587 for the number of years purchaſe required agreeable to Mr. *De Moivre's* hypotheſis.

The ſame value by the *Northampton* Table and the approximations in Problems 2d and 3d is 6.5.

EXAMPLE FOURTH.

Let the age of B be 30, that of C 20, and that of A 60 years: Then ſubſtituting A for C and *vice verſâ*, the firſt value by proceeding as in Caſe 2d will be found by Mr. *De Moivre*'s hypotheſis equal to 1.85, which added to 6.882, the value of the three joint lives by the ſame hypotheſis, and the *ſum* ſubtracted from 14.684 (the value of the life of B by Table 5th) gives 5.952 for the anſwer.

By the *Northampton* Table and the approximations in Problems 2d and 3d, this Reverſion will be found worth 5.985 years purchaſe.

Had the age of all the three lives been 60, the value by Mr. *De Moivre*'s hypotheſis would have been equal to 1.992; by the *Northampton* Table and the approximations juſt mentioned, it would have been equal to 2.055; and by the ſame Table of Obſervations (taking the values of the ſingle and joint lives from Table 7th, Appendix, and the two Tables in pag. 64 and 74) the *exact* value would have been equal to 2.118.

COROL-

COROLLARY.

From the preceding Solution the value of the life of B after A, provided A ſhould die *after* C, may be eaſily determined, the rule being as follows:

" Find by this Problem the value of an " Annuity on the life of B after A, pro- " vided A ſhould die *before* C. Find next " the value of the joint lives of A and B. " Then will the *ſum* of theſe two values " ſubtracted from the value of the life of " B, be the number of years purchaſe re- " quired."

EXAMPLE.

Let the age of B be 20, that of A 30, and that of C 60 years.—By Example 2d of this Problem it appears that the firſt value, according to Mr. *De Moivre*'s hypotheſis, is 2.422, and by Table 6th that the ſecond value is 11.711. Their *ſum*, therefore, or 14.133, ſubtracted from 15.891 (the value of the life of B by Table 5th) leaves 1.758 for the anſwer.

The ſame value by the *Northampton* Table and the approximations in Problems 2d and 3d is 1.711.

PROB-

PROBLEM XXVIII.

To find the preſent worth of an eſtate of a given yearly value to be entered upon at the deceaſe of A and B, provided C ſhould ſurvive one life in particular (A.)

SOLUTION.

Find the value of the given eſtate by Problem 16th if C ſhould live till the deceaſe of A.—Find next by Problem 27th the value of the life of B after A, provided C ſhould ſurvive A. Subtract the laſt value from the former, and the remainder will be the anſwer.

The value in *annual* payments till the claim *is determined*, is the ſingle payment divided by the value of the joint lives A and C.—And the value in *annual* payments till the claim *becomes due* (provided it ſhould be determined in favour of the purchaſer) is the ſingle payment divided by the value of the joint lives A and C, increaſed by the value of the life of B after A, provided C ſhall have ſurvived A (found by Problem 27th.)

If the value of the *eſtate* in *one* payment be divided by 1 *l.* increaſed by its intereſt for a year, the quotient will be the value of an equivalent *ſum*.

EXAMPLE.

Let the age of A be 30 years, that of B 60, and that of C 20.—The eſtate 40 *l. per annum*.

The firſt value by Problem 16th, agreeable to the hypotheſis, is 7.427 years purchaſe. The ſecond value by Problem 27th and the hypotheſis, is 1.235 years purchaſe, which being ſubtracted from 7.427, and the remainder, or 6.192, multiplied by 40, we have 247.68 *l.* for the anſwer in *one preſent* payment.

And 247.68 *l.* divided by 1.04 (or 1 *l.* increaſed by its intereſt for a year) quotes 238.15 *l.* for the value of an equivalent *ſum* (or of 1000 *l.*) in *one* payment.

The value of the joint lives of A and C (by Table 6th) is 11.711, and 238.15 *l.* divided by this quantity quotes 20.335 *l.* for the value of the above ſum in *annual* payments to be made at the end of every year till the claim is *determined*.——But if thoſe payments are to be continued in the ſame manner till the claim *becomes due*, the diviſor 11.711 muſt be increaſed by 1.235

(found by Problem 27th) which will conſequently reduce them to 18.395 *l.*

The value of the *eſtate* in one payment by the *Northampton* Table, and the approximations in Problems 2d and 3d, may be found equal to 248.52 *l.* and of the *equivalent* ſum equal to 238.96 *l.*

PROBLEM XXIX.

To find the value of a given ſum payable at the death of C, provided A ſhould be the *firſt*, B the *ſecond*, and C the *third* that fails of the three lives A, B, and C.

SOLUTION.

CASE 1ſt. *If the life of C is the oldeſt of the three.* From the perpetuity ſubtract the value of an Annuity upon the longeſt of three equal lives whoſe common age is that of C, and take $\frac{1}{6}$th the remainder.—Then it will be, as the product of the complements* of the two other lives is to the ſquare of the complement of C, ſo is the ſaid $\frac{1}{6}$th remainder to a fourth proportional, which will be the number of years purchaſe an equivalent *eſtate* is worth; and this being multiplied into the given ſum, and the product divided by the perpetuity increaſed by unity, the *quotient* will be the anſwer.

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* The complement is equal to twice the expectation.

CASE 2d. *If the life of B is the oldeſt.* From the perpetuity ſubtract the value of an Annuity upon the longeſt of three equal lives whoſe common age is that of B, and take $\frac{1}{6}$th the remainder.——Multiply half the perpetuity by 1 *l.* payable at the end of as many years as are expreſſed by the complement of C. Then it will be, as the complements of A and C multiplied into each other are to the ſquare of the complement of B, ſo is the *difference* between the product juſt mentioned and the $\frac{1}{6}$th remainder to a fourth proportional, which muſt be *reſerved.* Find next by Problem 4th the value of an Annuity on the life of C for a term equal to the difference between his complement and that of B, which ſubtract from the perpetuity and take $\frac{1}{2}$ the remainder. Then it will be, as the expectation of A is to the expectation of B, ſo is the ſaid remainder to a fourth proportional; which being multiplied into the value of 1 *l.* payable at the end of a term equal to the complement of B, and added to the *reſerved* proportional, the *ſum* will be the number of years purchaſe an equivalent *eſtate* is worth, when the $\frac{1}{6}$th remainder is *greater* than the product; but if *leſs*, the reſerved proportional muſt be ſubtracted, and the *remainder* will be the number of years purchaſe the ſaid eſtate is worth.—The value of the given *ſum* is obtained from hence, in this and the following Caſes, as in Caſe 1ſt.

CASE 3d. *If A is the oldeſt and B the youngeſt.* From the perpetuity ſubtract the value of an Annuity upon the longeſt of three lives whoſe common age is that of A, and take $\frac{1}{6}$th the remainder.—Then it will be, as the complements of B and C multiplied into each other are to the ſquare of the complement of A, ſo is the ſaid $\frac{1}{6}$th remainder to a fourth proportional, which muſt be *reſerved.*—Find the value of an Annuity on the life of C for as many years as are expreſſed by the difference between his complement and that of A, which being ſubtracted from the perpetuity, let the remainder be multiplied into the expectation of A.—Find next by Problem 4th, Corollary 1ſt, the value of an Annuity during the term juſt mentioned upon the longeſt of two equal lives whoſe common age is that of C, which ſubtract from the perpetuity, and multiply the remainder by the expectation of C. Add theſe two products together; then it will be, as the complement of B is to the value of 1 *l.* payable at the end of as many years as are expreſſed by the complement of A, ſo is the *ſum* of the two products to a fourth proportional, which muſt alſo be *reſerved.*——Laſtly, multiply the perpetuity into the expectation of A, and alſo into the value of 1 *l.* payable at the end of as many years as are expreſſed by the complement of C: divide the product by the expectation of B; then the quotient ſubtracted from the ſum of the two *reſerved* proportionals will give the number of years purchaſe required.

CASE 4th. *If A is oldeſt and C youngeſt.* Find by Caſe 3d (after ſubſtituting there B for C, and C for B) the value of the given eſtate upon the ſuppoſition that *B* is the youngeſt.—Then find the value of an Annuity on the life of C for as many years as his complement exceeds that of B: Subtract this value from the perpetuity, let the remainder be multiplied into 1 *l.* payable at the end of a term equal to the complement of B; add this product to the value found by Caſe 3d, and *reſerve* the ſum.—Laſtly, multiply the perpetuity into the value of 1 *l.* payable at the end of as many years as are expreſſed by the complement of C, and alſo into the expectation of B; divide the product by the expectation of C, and this quotient being ſubtracted from the *reſerved* ſum, the remainder will be the number of years purchaſe required.

In either of thoſe caſes the value in *annual* payments, to be continued till the claim is *determined*, may be obtained by dividing the value in *one* payment by an Annuity on the joint lives of B and C.

If the *annual* payments are to be continued till the claim *becomes due*; that is, till the claim is determined, and afterwards till it becomes payable provided it ſhould be determined in favour of the purchaſer, divide the value in *one* payment by the value of the joint lives B and C added to the value (found by Corollary to Problem 27th) of an Annuity on the life of C after B provided B ſurvives A.

EXAMPLE FIRST*.

Let the age of A be 20 years, that of B 30, and that of C 60.—The ſum 1000 *l.*— This

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[*] The anſwers to theſe four Examples are computed agreeable to Mr. *De Moivre's* hypotheſis, and are compared in Note H of the Appendix not only with the ſame

This Example, it is evident, belongs to Caſe 1ſt.—The value of an Annuity upon the longeſt of three equal lives aged 60 (by Problem 6th) is 12.987, which ſubtracted from 25 (the perpetuity) and the remainder divided by 6 becomes equal to 2.002. Therefore as 56 (the complement of B) multiplied into 66 (the complement of A,) or 3696, is to 676 (the ſquare of 26, the complement of C,) ſo is 2.002 to .3661, which multiplied into 1000 (the given ſum) and divided by 26 (the perpetuity increaſed by unity) quotes 14.08*l.* for the anſwer in *one preſent* payment.

The value of the joint lives B and C (by Table 6th) is 7.779; hence 14.08*l.* divided by this value gives 1.81 *l.* for the *annual* payments till the claim *is determined* *.—The value of the life of C after B, provided

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ſame values in *exact* conformity to the *Northampton* Table of Obſervations, and alſo with the values calculated from this Table and the rules in Problems 2d and 3d for approximating to the ſingle and joint lives; but likewiſe with the values as they are derived from a different inveſtigation of the Problem.

[*] The *annual* payments in thoſe Examples are ſuppoſed to be made at the *end* of every year. If they are to be made at the *beginning* of each year, unity muſt be added to the number by which the ſingle payment is divided.

B ſurvives A, may be found by Corollary to Problem 27th equal to .11, which being added to 7.779 and 14.08*l.* divided by their ſum, we ſhall have 1.784*l.* for the value in *annual* payments till the claim *becomes due*.

EXAMPLE SECOND.

Let the age of A be 20, that of B 60, and that of C 30 years.—The ſum 1000*l.*—By following the ſteps directed in the firſt part of Caſe 2d the $\frac{1}{6}$th remainder will be found equal to 2.002. Half the perpetuity, or 12.5, multiplied into .111, the value of 1*l.* (by Table 1ſt) at the end of 56 years, is equal to 1.3875. As, therefore, 3696 is to 676 ſo is .6145 (the difference between 2.002 and 1.3875) to .1123, which muſt be *reſerved*.—The value of an Annuity on the life of C for $(\overline{56-26}=)$ 30 years will be found (by Problem 4th) equal to 13.393, which ſubtracted from 25 and divided by 2 quotes 5.803. Hence as 33 (the expectation of A) is to 13 (the expectation of B) ſo is 5.803 to 2.286, which multiplied into .3607, the value of 1*l.* (by Table 1ſt) at the end of 26 years, is reduced to .8246; and this added to .1123, *reſerved* above, gives .9369 for the number of years purchaſe an eſtate is worth. Conſequently the value of the propoſed ſum in *one preſent* payment is 36.035 *l.*; which divided by 7.779 (the value of the joint lives of B and C by Table 6th) quotes 4.632 *l.* for the value in *annual* payments till the claim is *determined.*—And 36.035*l.* divided by 7.779 added to .953 (the value of the life of C after B provided B ſurvives A by Corollary to Problem 27th) quotes 4.127 *l.* for the annual payments till the claim *becomes due.*

EXAMPLE THIRD.

Let the age of A be 60, that of B 20, and that of C 30 years. The ſum 1000 *l.*—By proceeding agreeable to the Solution in Caſe 3d, the firſt proportional to be *reſerved* appears to be .3661.—The value of C's life for 30 years is 13.393, which ſubtracted from 25, and then multiplied by 13 (the expectation of A) is equal to 150.891.—The value of the longeſt of two equal lives aged 30 for 30 years will be found (by Corollary 1ſt to Problem 4th) equal to 16.025, which ſubtracted from 25 and multiplied by 28 (the expectation of C) is equal to 251.3.—Therefore, as 66 (the complement of B) is to .3607 (the value of 1 *l.* by Table 1

1ſt, at the end of 26 years) ſo is 402.191 (the ſum of 150.891 and 251.3) to 2.198, which is the ſecond proportional to be *reſerved.* Laſtly, 25 multiplied into .111 (the value of 1*l.* at the end of 56 years) and alſo into 13 (the expectation of A) produces 36.075, which divided by 33 (the expectation of B) is reduced to 1.0931; and this quotient ſubtracted from 2.5641 (the ſum of the two *reſerved* proportionals) leaves 1.471 for the number of years purchaſe an *eſtate* is worth. Conſequently 56.577 *l.* is the value of the given ſum in *one preſent* payment; which being divided by 11.7, the value of the joint lives B and C by Table 6th, quotes 4.835 *l.* for the value in *annual* payments till the claim is *determined.*—And 56.577 *l.* divided by 11.7 added to 1.12 (the value of the life of C after B provided B ſurvives A by Corollary to Problem 27th) quotes 4.413 *l.* for the *annual* payments till the claim *becomes due.*

EXAMPLE FOURTH.

Let the age of A be 60, that of B 30, and that of C 20 years. The ſum 1000 *l.*—By ſubſtituting C for B and *vice verſâ*, the preſent Example belongs to Caſe 3d, and the value of an *eſtate* on ſuch a ſuppoſition appears to be 1.471 years purchaſe.—Let now the ſymbols repreſent the ages as above, and it is converted into an example of the fourth Caſe.—The value of an Annuity (by Problem 4th) on the life of C for (66 − 56 =) 10 years is 7.474, which ſubtracted from 25, and multiplied into .111 (the value of 1 *l.* by Table 1ſt, at the end of 56 years) becomes equal to 1.9454; therefore the ſum to be *reſerved* is 3.4164.—The value of 1 *l.* at the end of 66 years (by Table 1ſt) is .075, which multiplied into 25 and alſo into 28 (the expectation of B) produces 54.5; and this divided by 33 (the expectation of C) quotes 1.6515. Conſequently 1.7649 is the number of years purchaſe an *eſtate* is worth, and 67.881 *l.* is the value of the given ſum in *one preſent* payment.

If 67.881 *l.* be divided by 11.7, the value of the joint lives B and C by Table 6th, we ſhall have 5.802 *l.* for the value in *annual* payments till the claim is *determined*.

And if it be divided by 11.7 added to 1.76 (the value of the life of C after B provided B ſurvives A by Corollary to Problem 27th) we ſhall have 5.043 *l.* for the value in *annual* payments till the claim *becomes due*.

PROB-

PROBLEM XXX.

B if he ſurvives A in a given number of years (not exceeding the complement of the oldeſt of the three lives,) is to receive a given ſum, in caſe C is then extinct; to determine the value of his expectation in preſent money,

SOLUTION.

Divide the ſum of the decrements in the Table of Obſervations from the age of C for the given term by the term; and by this quotient divide the number of the living in the Table at the age of C; and this ſecond quotient call the *complement* of the life of C *for the given term.*—Find in the ſame manner the complement of the life of A for the given term.

From the value of an Annuity on the life of B for the given term (found by Problem 4th) ſubtract the value of the joint lives of B and C for the ſame term (found by Corollary 1ſt to Problem 4th,) Multiply the remainder by the given ſum, and divide the product by the complement of A *for the given term,* reſerving the quotient.—Multiply the given ſum by the value of an Annuity on the life of B for the given term and divide the product by double the complement of A *for the given term* multiplied into the complement of C *for the given term*; and the quotient ſubtracted from the laſt *reſerved* quotient will be the anſwer *.

EXAMPLE.

Let the age of A be 40, that of B 30, and that of C 50, the term 16 years, the ſum 1000 *l.* and the Table of Obſervations the *Northampton* Table, or Table 3d, Appendix. The ſum of the decrements in the Table for 16 years from 50, or the age of C, is 129, which divided by 16 (the given term) quotes 8.06. The number living at 50 is 284, which, divided by 8.06, gives 35.23 for the *complement* of the life of C for the given term. —In like manner it may be found that the *complement* of A for the given term is 45.

By Problem 2d the value of the life of B is 14.79, and of a life 16 years older 12.088; and conſequently (by Problem 4th) the value of the life of B for 16 years is 10.086.— In the ſame manner, the value of the joint lives of B and C (by Problem 3d and Corollary 1ſt to Problem 4th) for 16 years may be found to be 7.990.—The difference between theſe two values (or 2.096) multiplied into 1000 and divided by 45 (the complement of A for 16 years) gives 46.577 for the quotient to be *reſerved*.—The ſum 1000 *l.* multiplied by 10.086, and the product divided by twice the product of 45 into 35.23 gives .318; which ſubtracted from 46.577 leaves 46.259 *l.* the anſwer [*].

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* See what is ſaid of the Solution of this and the following Problems in note I, Appendix.

N. B. In

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[*] The ſolutions of this and all the following Problems are confined to a term of years not exceeding the complement, for the whole duration, of the oldeſt life. Theſe, however, are abundantly ſufficient, as it is never likely to happen that the values will be required for a longer term; and the ſolutions become ſo extremely complicated when they are extended further, that I have entirely omitted them.

The values, likewiſe, are only given in *one* preſent payment, being apprehenſive of rendering the work too tedious by giving them in *annual* payments; eſpecially as they may be ſo eaſily obtained from the ſolutions of the correſponding values for the *whole continuance* of the lives, where no other variation is neceſſary, than to find the values of the Annuities (by which the ſingle payments are to be divided) for the *given term*, inſtead of finding them for the *whole duration* of life.—Thus in the preſent caſe; it appears from the ſolution of Problem 18th that the value in *annual* payments is found by dividing the ſingle payment by an Annuity on the joint lives of A and B for 16 years. The value of this Annuity (by Corollary 1ſt to Problem 4th and the *Northampton* Table of

N. B. In this Example the values of the ſingle and joint lives have been calculated by the rules in Problems 2d and 3d, taking the complements of the lives in ſtrict agreement to the *Northampton* Table; or 56.76 and 35.88 for the ages of 30 and 50, and 39.74 and 20.54 for the ages of 46 and 66.

PROBLEM XXXI.

To find the value of a given ſum payable if A ſhould be the *firſt* that fails of the three lives A, B, and C, in a given number of years, not exceeding the complement of the oldeſt life.

SOLUTION.

Find firſt, in the manner directed in the laſt Problem, the *complements* of the lives of A, B, and C for the *given term.*—Then, from the perpetuity ſubtract the value of the longeſt of two equal lives whoſe common complement, for their whole duration, is the given term (found agreeable to the hypotheſis

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of Obſervations) is 8.476; therefore the value of the given ſum in *annual* payments (to be made at the end of each year) is 5.457 *l.*

In general, the values of thoſe Reverſions for terms are ſo inconſiderable in *ſingle* payments, that it can hardly be worth while to diminiſh them by *annual* ones.

pothefis of an equal decrement.) Multiply *half* the remainder into the ſum of the complements of B and C for the given term, and *reſerve* the product.—Subtract the value (found by Problem 6th and agreeable to the hypotheſis of an equal decrement) of the longeſt of three equal lives, whoſe common complement for the whole duration is the given term, from the perpetuity.—Multiply *one third* of the remainder into the given term, and let this product alſo be *reſerved.*—Then ſay, as the complements for the given term of A, B, and C multiplied into each other are to the ſquare of the given term, ſo is the *difference* between the two *reſerved* products to a fourth proportional; which being multiplied into the given ſum, divide the product by the perpetuity increaſed by unity, and *reſerve* the quotient.—Find by Problem 9th the value of the given ſum at the death of A, ſhould that happen in the given term; ſubtract the *reſerved* quotient from this value, and the remainder will be the anſwer.

EXAMPLE.

Let the age of A be 30, that of B 40, and that of C 50.—The term 16 years, and the propoſed ſum 1000 *l.*—By the rule in page 55 the value of the longeſt of two equal lives whoſe complement is 16, (or whoſe age by *De Moivre*'s hypotheſis is 70) is 8.026; which ſubtracted from 25 (the perpetuity) and half the remainder, or 8.487, multiplied into 80.23 (the ſum of the complements of B and C for the given term by Table 3d) gives 680.91 for the product to be *reſerved.*—By Problem 6th and Tables 5th and 6th, the value of the longeſt of three equal lives whoſe common complement is 16, or age 70, is 8.971. This being ſubtracted from 25 (the perpetuity) and divided by 3 quotes 5.343, which multiplied into 16 gives 85.488 for the ſecond product to be *reſerved.*—Hence, as 93.504 * (the product of the three complements of A, B, and C for the given term) is to 256 (the ſquare of 16) ſo is 595.43 (the difference between the two *reſerved* products) to 1.630: which being multiplied into 1000, and divided by 26 (the perpetuity increaſed by unity) gives 62.70 for the quotient to be *reſerved.*—By Problem 9th the value, by the *Northampton* Table, of the given ſum at the death of A, ſhould that happen in 16 years, is 195.7, which being leſſened by 62.70, we have 133 *l.* for the value required in *one preſent* payment.

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* The complements of the lives of A, B, and C for 16 years are (by Table 3d) reſpectively equal to 58.98, —45,—and 35.23.

SCHOLIUM.

In finding the values of the longeſt of two or three equal lives, or of three joint lives, it is neceſſary to be poſſeſſed of the values of ſingle lives and *two* joint lives; and theſe values, whenever neceſſary to be obtained for that purpoſe in ſolving the preceding Problem and all that follow, ſhould be taken either from Tables of the values of ſingle and joint lives by Mr. *De Moivre*'s hypothesis, or computed agreeable to the hypotheſis of an equal decrement, taking the term for the complement, by the rules in Problems 2d and 3d.—When not neceſſary to be obtained for the purpoſe of finding the values of two or three *equal* lives in ſolving theſe Problems, they ſhould be taken agreeable to the Table of Obſervations uſed (or as nearly ſo as poſſible from the rules in Problems 2d and 3d) taking double the expectation (found by Problem 1ſt) for the complement.

PROBLEM XXXII.

To find the value of a given ſum payable at the death of A, ſhould his life be the *ſecond* that fails of the three lives A, B, and C, in a given number of years, not exceeding the complement of the oldeſt life.

SOLUTION.

Firſt find the complements of the lives of A, B, and C for the given term as directed in Problem 30th.——From the perpetuity ſubtract the value of an Annuity on the longeſt of two equal lives whoſe common complement for their whole duration is the given term. Multiply *half* the remainder into the ſum of the complements of B and C (juſt found) and *reſerve* the product.—Subtract the value of an Annuity on the longeſt of three equal lives, whoſe common complement for their whole duration is the given term, from the perpetuity: multiply *one third* of the remainder into twice the given term, and let this product alſo be *reſerved.* —Then ſay, as the complements for the given term of the three lives A, B, and C, multiplied by one another, are to the ſquare of the given term, ſo is the *difference* of the two *reſerved* products to a fourth proportional, which being multiplied into the given ſum, and divided by the perpetuity increaſed by unity, will give the value ſought.

EXAMPLE.

Let the age of A be 30, that of B 40, and that of C 50. The term 16 years, and the propoſed ſum 1000*l.*—By proceeding in the ſame manner as in the Example to Problem 31ſt, the firſt product to be *reſerved* will be found equal to 680.91, and the ſecond product to be *reſerved* equal to 170.976. Therefore as 93504, the three complements for the given term (by Table 3d) multiplied into each other, is to 256 (the ſquare of 16) ſo is 509.934 (the *difference* of the two *reſerved* products) to 1.396; which being multiplied into 1000, and then divided by 26 (the perpetuity increaſed by unity) gives 53.7 *l.* for the anſwer in *one preſent* payment.

PROBLEM XXXIII.

To find the value of a given ſum payable at the death of A, ſhould his life be the *laſt* that fails of the three lives A, B, and C, in a given number of years not exceeding the complement of the oldeſt life.

SOLUTION.

Find the complements of the lives of A, B, and C for the given term as directed in Problem 30th.—From the perpetuity ſubtract the value of an Annuity on the longeſt of three equal lives whoſe common complement for their whole duration is the given term, and take *one third* of the remainder: Then it will be, as the product ariſing from the complements, for the given term, of A, B, and C multiplied into each other, is to the cube of the given term, ſo is the *one third* remainder to a fourth proportional; which being multiplied into the given ſum, and then divided by the perpetuity increaſed by unity, will give the value required.

EXAMPLE.

Let the age of A be 30, that of B 40, and that of C 50. The term 16 years. The ſum 1000 *l.*—By Problem 6th and *De Moivre*'s hypotheſis, the value of three equal lives aged 70 is 8.971, which ſubtracted from 25 (the perpetuity) and the remainder divided by 3 gives 5.343. Therefore as 93504 (the product of the complements of A, B, and C for the given term by Table 3d) is to 4096 (the cube of 16) ſo is 5.343 to .234. Conſequently 9 *l.* is the value in *one preſent* payment.

PROBLEM XXXIV.

To find the value of a given ſum payable at the extinction of the lives of A and B, ſhould they be the *firſt* that will fail of the three lives A, B, and C, in a given number of years not exceeding the complement of the oldeſt life.

SOLUTION.

Find the complements of the lives of A, B and C for the given term as directed in Problem 30th. Find next by the rule in page 55 the value of the longeſt of two equal lives whoſe common complement for their whole duration is the given term. Subtract this value from the perpetuity; then it will be, as the product, ariſing from the complements of the three lives for the given term multiplied by one another, is to this remainder, ſo is the ſquare of the given term to a fourth proportional, which muſt be *reſerved.*—Again; find by Problem 6th the value of the longeſt of three equal lives whoſe common complement for the whole duration is the given term. Subtract this value from the perpetuity, and take *two thirds* of the remainder. Then ſay, as the product, ariſing from the three complements for the given term multiplied into each other, is to the *two third* remainder, ſo is the cube of the given term to a fourth proportional, which being ſubtracted from the *reſerved* proportional, let the remainder be multiplied into the given ſum; and this product. divided by the perpetuity increaſed by unity, will give the value required.

EXAMPLE.

Let the age of A be 30, that of B 40, and that of C 50.—The term 16 years, and the propoſed ſum 1000 *l.*——The value of two equal lives aged 70 (or whoſe common complement, by the hypotheſis, for the whole duration is 16) may be found equal to 8.026, which being ſubtracted from 25 (the perpetuity) leaves 16.974. Therefore as 2654 (the product of the complements of A and B for the given term by Table 3d) is to 16.974, ſo is 256 (the ſquare of 16) to 1.633, the proportional to be *reſerved.*——

The value of the longeſt of three equal lives aged 70 is 8.971, which ſubtracted from 25, and the remainder multiplied into $\frac{2}{3}$, gives 10.686. Hence as 93504 (the product of the three complements for the given term into each other) is to 10.686, ſo is 4096 (the cube of 16) to .468.—And .468 ſubtracted from 1.633 (the *reſerved* proportional) leaves 1.165; which being multiplied into 1000, and then divided by 26, quotes 44.808*l.* for the anſwer in *one preſent* payment.

PROBLEM XXXV.

To find the value of a given ſum payable at the death of A and B, ſhould their lives be the *laſt* that will fail of the three lives A, B, and C, in a given number of years not exceeding the complement of the oldeſt life.

SOLUTION.

This Problem reſtricted, as above, to a number of years not exceeding the complement of the oldeſt life, is anſwered in the ſame manner with Problem 33d.

PROBLEM XXXVI.

To find the value of a given ſum payable at the death of A, ſhould his life be the *firſt* or *ſecond* that ſhall fail of the three lives A, B, and C in a given number of years not exceeding the complement of the oldeſt life.

SOLUTION.

Find by Problem 33d the value of the given ſum payable if A ſhould be the *laſt* that dies of the three lives in the given term. Subtract this from the value of the given ſum (found by Problem 9th) payable at the death of A ſhould that happen in the given term; and the remainder will be the anſwer.

EXAMPLE.

Suppoſe the age of A to be 30, that of B 25, and that of C 15.—The term 10 years, and the propoſed ſum 1000*l*.—The firſt value by Problem 33d may be found equal to .815, and the ſecond value by Problem 9th and the *Northampton* Table, equal to 130.115. Their difference therefore, or 129.3 *l*, is the anſwer in *one preſent* payment.

PROB-

PROBLEM XXXVII.

To find the value of a given ſum payable at the death of A, if his life ſhould *not* be the ſecond that fails; that is, if it ſhould be the *firſt* or *laſt* that ſhall fail of the three lives A, B, and C in a given number of years not exceeding the complement of the oldeſt life.

SOLUTION.

Find by Problem 32d the value of the given ſum provided A ſhould be the *ſecond* that dies in the given term; ſubtract this value from the value of the given ſum (by Problem 9th) payable at the death of A ſhould that happen in the given time, and the remainder will be the anſwer.

EXAMPLE.

Let the age of A be 30, that of B 25, and that of C 15. The term 10 years, and the propoſed ſum 1000 *l.*—The firſt value (by Problem 32d) may be found equal to 16.065; and the ſecond value (by Problem 9th and the *Northampton* Table) equal to 130.115. Conſequently 114.05 *l.* is the anſwer in *one preſent* payment.

PROBLEM XXXVIII.

To find the value of a given ſum payable at the death of A, if his life ſhould *not* be the firſt that fails; that is, if it ſhould be the *ſecond* or *third* that ſhall fail of the three lives A, B, and C in a given number of years not exceeding the complement of the oldeſt life.

SOLUTION.

Find by Problem 31ſt the value of the given ſum if A ſhould be the *firſt* that dies in the given time. Subtract this from the value of the ſum (by Problem 9th) payable at the death of A, ſhould that happen in the given time; and the remainder will be the anſwer.

EXAMPLE.

Suppoſe the age of A to be 30, that of B 25, and that of C 15. The term 10 years, and the propoſed ſum 1000 *l.*—The firſt value (by Problem 31ſt) may be found equal to 113.23; and the ſecond value (by Problem 9th and the *Northampton* Table) equal to 130.115. Their difference therefore,

fore, or 16.885 *l*, is the value required in *one preſent* payment.

PROBLEM XXXIX.

To find the value of a given ſum payable at the death of C, provided A ſhould be the firſt, B the ſecond, and C the third that fails of the three lives A, B, and C, in a given number of years not exceeding the complement of the oldeſt life.

SOLUTION.

Find by Problem 33d the value of the given ſum payable at the death of C, ſhould his life be the laſt that fails of the three lives in the given time; and *half* this value will be the anſwer.

EXAMPLE.

Let the age of A be 50, that of B 40, and that of C 30. The term 16 years, and the propoſed ſum 1000 *l*.—By Problem 33d the value of the given ſum at the death of C, ſhould his life be the laſt that fails in the given time is 9 *l*.—Conſequently 4.5 *l*. is the value required in the preſent caſe in *one* payment.

PROB-

PROBLEM XL.

To find the value of an Annuity on the life of B *after* A, on the particular condition that A's life when it fails ſhall fail *after* the life of C, and alſo in a given term of years not exceeding the complement of the oldeſt life.

SOLUTION.

Find the complements of A and C for the given term as directed in Problem 30th.—Let D repreſent the life whoſe complement is the given term. Then find (by Problem 6th) the value (on the ſuppoſition of an equal decrement of life) of the three lives B, D, and D, taking double the expectation of B, according to the Table of Obſervations, for its complement *. Find alſo by the rule in page 55 the value of the longeſt of two equal lives whoſe complement is that of D. Multiply *half* the difference between thoſe values into the ſquare of the given term; and the product, divided by the complements of A and C for the given term drawn into each other, will give the number of years purchaſe required.

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* In all the following Examples the *expectation* of life is taken from Table 4th, and the complements for the given term are computed from Table 3d, Appendix.

EXAMPLE.

Let the age of A be 40, that of B 30, and that of C 50. The term 16 years.—In this caſe the age of D is 70.—By Problem 6th and under the reſtrictions mentioned above, the value of an Annuity on the longeſt of three lives aged 70, 30, and 70 is 15.724. By the rule in page 55 the value of the longeſt of two equal lives aged 70 is 8.026: *half* the difference, therefore, between theſe two values is 3.589, which being multiplied into 256 (the ſquare of 16) and then divided by 1585.35 (the product of the complements of A and C for the given term into each other) gives .579 for the number of years purchaſe required.

COROLLARY.

If the Annuity *after* the Survivorſhip mentioned in this Problem, is to be continued on the life of B only for the *remainder of the given term*, inſtead of the *whole duration* of the ſaid life; the value will be determined as follows:

" From the value of an Annuity certain
" for the given term (found by Table 2d)
" ſubtract the value of an Annuity on the
" longeſt of two equal lives of the age of
" D (found by the rule in page 55) and
" *reſerve* the remainder.—Multiply the dif-
" ference, between the value of an Annuity
" certain for the given term and the value of
" an Annuity on the longeſt of three equal
" lives of the age of D, (found by Problem
" 6th) into the given term. Divide the pro-
" duct by the complement of B for the given
" term, and take *half* the difference between
" this quotient and the *reſerved* remainder.
" —Then ſay, as the complements of A and
" C for the given term drawn into each
" other are to this *half* difference, ſo is the
" ſquare of the given term to a fourth pro-
" portional, which will be the anſwer."

EXAMPLE.

Let the ages of A, B, and C be reſpectively 40, 30, and 50.—And let the term alſo be 16 years. The value of an Annuity certain for 16 years is 11.652. The value of the longeſt of two equal lives aged 70 is 8.026. The remainder to be *reſerved*, therefore, is 3.626.—The value of the longeſt of three equal lives aged 70 is 8.971, which ſubtracted from 11.652 leaves 2.681. This difference being multiplied into 16 (the given term) and then divided by 58.98 (the complement of B for the given term) quotes .727; which again ſubtracted from 3.626, *reſerved* above, and the remainder divided by 2, we have 1.449.—Hence as 1585.35 (the complements for the given term of A and C drawn into each other) is to 1.449 ſo is 256 (the ſquare of 16) to .234, the number of years purchaſe required.

PROBLEM XLI.

To find the value of an Annuity on the life of B after A, on the particular condition that A's life when it fails, ſhall fail *before* the life of C, and alſo in a given number of years not exceeding the complement of the oldeſt life.

SOLUTION.

Find by Problem 40th the value of an Annuity on the life of B after A, on condition that A's life ſhall fail *after* C within the given term.——Let D, as before, repreſent a life whoſe decrements are equal, and whoſe complement for the whole duration is the given term; then ſubtract the value of the two joint lives B and D (by Problem 3d) taking double the expectation of B by the Table of Obſervations for its complement, from the value of an Annuity on the ſingle life of B. Multiply the remainder into the given term, and divide the product by the complement of A for the term (found by the rule in Problem 30th.) —The difference between this quotient and the value firſt found will be the number of years purchaſe required.

EXAMPLE.

Suppoſe the age of A to be 40, that of B 30, and that of C 50. The term 16 years. The firſt value, by Problem 40th, appears to be .579.—By Problem 3d the value of the joint lives D and B (the age of D being in this caſe 70) is 5.55, which ſubtracted from 14.79 (the value of the life of B by Problem 2d) leaves 9.24. This remainder being multiplied into 16 (the given term) and then divided by 45 (the complement of A for the given term) quotes 3.285. Conſequently 2.706 is the number of years purchaſe required.

COROL-

COROLLARY.

When the Annuity is to be continued on the life of B only during *the remainder of the given term* after the Survivorſhip propoſed in this Problem, inſtead of being continued during the *whole* of the ſaid life, the value will be determined as follows:

" From the value of an Annuity on the " life of B for the given term (found by " Problem 4th) ſubtract the value of an " Annuity on the joint lives of A and B " for the given term (found by Corollary " 1ſt to Problem 4th) and *reſerve* the re- " mainder.——Find next by the Corollary " to Problem 40th the value of an Annuity " on the life of B after A on condition that " the life of A ſhall fail *after* C within the " given term; which value being ſubtracted " from the *reſerved* remainder will give the " anſwer in this caſe."

EXAMPLE.

The ages of A, B, and C being reſpectively 40, 30, and 50, let the term be 16 years.—The value of the joint lives of A and B for 16 years (by Corollary 1ſt, Problem 4th, and taking double the expectations of the lives by the *Northampton* Table for their complements) is 8.476; and the value of the ſingle life of B for the ſame term (by Problem 4th) is 10.072. Therefore 1.596 is the remainder to be *reſerved.*—By the Corollary to Problem 40th the ſecond value appears to be .234; which being ſubtracted from 1.596, leaves 1.362 for the number of years purchaſe required.

PROBLEM XLII.

To find the preſent worth of an eſtate or Annuity to be entered upon at the deceaſe of A and B, provided both of them ſhall die in a given term not exceeding the complement of the oldeſt of the three lives, and that C ſhall ſurvive one of them in particular (A).

SOLUTION.

Find the complements of A, B, and C for the given term as directed in Problem 30th, and let D, as in the two preceding Problems, repreſent a life whoſe complement for the whole duration is the given term, and whoſe decrements are equal.—Then find by the rule in page 55 the value of an Annuity on the longeſt of two equal lives of the age of D, which value being ſubtracted from the perpetuity, multiply the remainder into the ſquare of the given term; and dividing the product by the complements of A and B for the given term drawn into each other, let the quotient be *reſerved.*

Find next by Problem 6th the value of an Annuity on the longeſt of three equal lives of the age of D; ſubtract this value from the perpetuity, and take *half* the remainder. Then ſay; as the complements of A, B, and C for the given term multiplied into each other, are to this *half* remainder, ſo is the cube of the given term to a fourth proportional; which being ſubtracted from the *reſerved* quotient will give the anſwer *.

EXAMPLE.

Let the age of A be 40, that of B 30, and that of C 50. The term 16 years.— Here the age of D being 70, the value of an Annuity on the longeſt of two ſuch equal lives is 8.026.—The difference between this value and 25 (the perpetuity) being multiplied into 256 (the ſquare of 16) becomes equal to 4345.344; and this product divided by 2614 (the complements of A and B for the given term drawn into each other) gives 1.662 for the quotient to be *reſerved.*—The value of the longeſt of three equal lives aged 70 is 8.971, and *half* the difference between this value and 25 (the perpetuity) is 8.014. Hence as 93504 (the product of the complements of A, B, and C for the given term) is to 8.014 ſo is 4096 (the cube of 16) to .351. And .351 ſubtracted from 1.662 (the *reſerved* quotient) leaves 1.311 for the number of years purchaſe required.

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* In this and the two preceding Problems it may be obſerved that the Solutions are confined to the values of reverſionary *eſtates* or *annuities*. But the values of reverſionary *ſums* are very eaſily obtained from hence, only "by multiplying the number of years purchaſe into the "given ſum, and then dividing the product by the per- "petuity increaſed by unity."

APPENDIX.

Note (A). *See Chap. 2d, Page* 57.

LET *r* be 1 *l.* increaſed by its intereſt for a year, *b* the number of the living in the Table oppoſite to the preſent age of the given life, and *c*, *d*, *e*, &c. the numbers of the living at the end of 1, 2, 3, &c. years from his preſent age. Then, from what has been ſaid in page 43, the value of an Annuity on this life will be $\frac{c}{br} + \frac{d}{br^2} + \frac{e}{br^3}$ &c. &c. —Call this value N.—Let *a* ſignify the number of the living in the Table oppoſite to the age of a perſon one year younger than the given life, and it follows that the value of an Annuity on the life of this younger perſon will be expreſſed by $\frac{b}{ar} + \frac{c}{qr^2} + \frac{d}{ar^3} + \frac{e}{ar^4}$, &c.——But this ſeries is equal to $\frac{b}{ar} \times \overline{1 + \frac{c}{br} + \frac{d}{br^2}, \&c.} = \frac{b}{ar} \times \overline{1 + N}$ agreeable to the theorem in page 57.

Note (B). *See Chap. 2d, Page* 58.

In the caſe of the *youngeſt* life, it is evident there is no other difference between the operations in this rule and the theorem in note (A), than that thoſe in the former are begun at the laſt year and continued upwards to the firſt year of life, and that thoſe in the latter are begun at the firſt year and continued downwards to the laſt year of life. Let $a, b, c, d, e,$ - - - - - - - - - z expreſs the numbers of perſons at all ages in the Table, n the difference between the ages of the youngeſt and oldeſt life, and r 1 *l.* increaſed by its intereſt for a year.—Then will the value of the youngeſt life by the theorem be $\frac{b}{ar}+\frac{c}{ar^2}+\frac{d}{ar^3}+\frac{e}{ar^4}$, &c. - - - - - $+\frac{z}{ar^n}$; and by the *Rule* the ſum of all the values of 1 *l.* payable if this life exiſts to the ſeveral ages of the oldeſt, 2d, 3d, 4th, &c. lives is $\frac{z}{ar^n}+\frac{y}{ar^{n-1}}+\frac{x}{ar^{n-2}}+\frac{u}{ar^{n-3}}$ - - - - - - - $+\frac{b}{ar}$.—The value of an Annuity found by the *Theorem* is, in the preſent inſtance, to be multiplied by unity, and will, therefore, remain unaltered, and be juſt the ſame with the value found by the *Rule*.

Again; the value of a life one year older by the *Theorem* is $\frac{c}{br}+\frac{d}{br^2}+\frac{e}{br^3}$ - - - - - - - - $+\frac{z}{br^{n-1}}$, which being multiplied by the value of 1 *l.* payable if the youngeſt life ſhould continue one year, or by $\frac{b}{ar}$, is equal to $\frac{c}{ar^2}+\frac{d}{ar^3}+\frac{e}{ar^4}$ - - - - - - - - $+$ $\frac{z}{ar^n}$. Now the ſum of the values of 1 *l.* payable if the youngeſt life ſhould exiſt to the age of the oldeſt and all the intermediate ages between it and a life one year older than the youngeſt, is expreſſed by the ſame fraction in a contrary order, or by $\frac{z}{ar^n}+\frac{y}{ar^{n-1}}+\frac{x}{ar^{n-2}}$ - - - - - - - $+\frac{c}{ar^2}$.—The value of a life two years older by the *Theorem* is $\frac{d}{cr}+\frac{e}{cr^2}+\frac{f}{cr^3}$ - - - - - - $+\frac{z}{cr^{n-2}}$, which being multiplied by 1 *l.* payable if the youngeſt life exiſts two years, or by $\frac{c}{ar^2}$, is equal to $\frac{d}{ar^3}+\frac{e}{ar^4}+\frac{f}{ar^5}$ - - - - - - - $+\frac{z}{ar^n}$; and this is eaſily determined to be the ſum of all the values of 1 *l.* payable if the youngeſt life exiſts to the age of the oldeſt and all the intermediate ages between it and a life two years older than the youngeſt.—Laſtly, the value of an Annuity on the oldeſt life but one is equal to $\frac{z}{yr}$, which being multiplied by the value of 1 *l.* payable if the youngeſt life exiſts to this age, or by $\frac{y}{ar^{n-1}}$, is equal to $\frac{z}{ar^n}$; and this is evidently the value of 1 *l.* payable if the youngeſt life exiſts to the age of the oldeſt.——From hence the truth of the rule is manifeſt.

Note (C). PROBLEM X. *Page* 117.

The truth of this ſolution will appear from the following demonſtration. Suppoſe the amount required of 56 or *(n)* Annuities, each 1 *l*, on 56 (or *n*) lives all aged 30.—In this caſe, as one life will drop every year by *De Moivre*'s hypotheſis †, the amount of the ſeveral Annuities will be as follows;

Annuity
1. - - - - 0
1. - - - - 1
1. - - - - $1 + r$ *.
1. - - - - $1 + r + r^2$.
1. - - - - $1 + r + r^2 + r^3$.
&c. &c. &c. &c. &c. to *n* terms.

Sum 56 - - $\overline{n-1}. + \overline{n-2}.r + \overline{n-3}.r^2 + \overline{n-4}.r^3$, &c. = the amount of all the Annuities, or of *n* Annuities on *n* lives aged 30.—Therefore the amount of one Annuity of 1 *l.* on one life aged 30 muſt be $\overline{n-1} + \overline{n-2}.\ r + \overline{n-3}.\ r^2$, &c. divided by *n*, or $\frac{n-1}{n} + \frac{n-2}{n} \times r + \frac{n-3}{n} \times r^2$, &c. to *n* terms.—In order to ſum up this ſeries it may be reſolved into the two following ones; $\overline{1 + r + r^2 + r^3 \cdots\cdots r^{n-1}}$ and $-\frac{1}{n} \times \overline{1 + 2r + 3r^2 + 4r^3 \cdots\cdots + nr^{n-1}}$. The firſt of theſe is eaſily found equal to $\frac{r^n - 1}{r-1}$, and the ſecond equal to $\frac{r^n}{r-1} - \frac{r^n-1}{n.\overline{r-1}|^2}$. Conſequently the whole amount of the Annuity will be expreſſed by $\frac{r^n-1}{n.\overline{r-1}|^2} - \frac{1}{r-1}$, which is the general rule.

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† See this hypotheſis explained in a note, page 3.

* *r* denotes 1 *l.* increaſed by its intereſt for a year.

Dr. *Price* in the Philoſophical Tranſactions (Vol. 66, No. 6.) has given the different methods of determining the values of Annuities certain, and Life-Annuities payable *yearly*, *half-yearly*, *quarterly*, and *momently*.——By reaſoning much in the ſame manner as he has done, the following theorems may be obtained for determining the *amounts* of Annuities laid up *yearly*, *half-yearly*, &c. either during a given number of years, or during the continuance of a given life.

Let r be the intereſt of 1 *l.* for a year, P the perpetuity, n the term of years during which the *annuity certain* is to accumulate, y the ſum to which this Annuity will amount when laid up *yearly*, h the ſame ſum when it is laid up *half yearly*, q the ſame ſum when it is laid up *quarterly*, and m the ſame ſum when it is laid up *momently*.——Let alſo Y, H, Q, and M denote the reſpective ſums to which the Annuity will amount when laid up yearly, half-yearly, quarterly, or momently *during the continuance of a life* whoſe complement is n.——Then

THEOREM I. - - - - $y = \frac{\overline{1+r}|^n}{r} - P$

THEOREM II. - - - $h = \frac{\overline{1+\frac{r}{2}}|^{2n}}{r} - P$

THEOREM III.---$q = \frac{\overline{1+\frac{r}{4}}\Big|^{4n}}{r} - P$

THEOREM IV.---$m = \frac{N}{r} - P$, where N ſtands for the number which has rn for its hyperbolic logarithm.

THEOREM V.---$Y = \frac{y}{nr} - P$

THEOREM VI.---$H = \frac{h}{nr} - P$

THEOREM VII.---$Q = \frac{q}{nr} - P$

THEOREM VIII.---$M = \frac{m}{nr} - P$

If n, or the term of years, be 50, and r, or the intereſt of 1 *l.* for a year, be .05, then the amount of an *Annuity certain* by Theorem 1ſt will be = 209.34 *l*, by Theorem 2d it will be = 216.28 *l*, by Theorem 3d it will be = 219.84 *l*, and by Theorem 4th it will be = 224 *l.*

Again; if n be the complement of a life whoſe age is 36, which by *De Moivre*'s hypotheſis is 50, and r, as before, be .05, then the amount of 1 *l.* *per annum* during this life will, by Theorem 5th, be = 63.73 *l*, by Theorem 6th = 66.49 *l*, by Theorem 7th = 67.93 *l*, and by Theorem 8th = 69.6 *l.*

Note (D). PROBLEM XI. *Page* 119.

Let V be the value of a life whoſe complement is n, v the value of 1 *l.* at the end of the firſt year, 2 *l.* at the end of the ſecond year, &c.; A the value of an Annuity certain for $\overline{n-1}$ years; G the value of two equal joint lives whoſe complement is n; and r, as uſual, 1 *l.* increaſed by its intereſt for a year. Then by reaſoning as in page 43 and ſuppoſing an equal decrement of life, $v = \frac{n-1}{nr} + \frac{2.\overline{n-2}}{nr^2} + \frac{3.\overline{n-3}}{nr^3}$, &c. $= n \times \overline{\frac{1}{nr} + \frac{2}{nr^2} + \frac{3}{nr^3}}$ &c. $- n \times \overline{\frac{1^2}{n.nr} + \frac{2^2}{n.nr^2} + \frac{3^2}{n.nr^3}}$, &c. But $\overline{\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3}}$, &c. $- \overline{\frac{1}{nr} + \frac{2}{nr^2} + \frac{3}{nr^3}}$, &c. $=$ V. Therefore $n \times \overline{A-V} = n \times \overline{\frac{1}{nr} + \frac{2}{nr^2} + \frac{3}{nr^3}}$, &c.----Alſo $\overline{\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3}}$, &c. $- \overline{\frac{2}{nr} + \frac{4}{nr^2} + \frac{6}{nr^3}}$, &c. $+ \overline{\frac{1^2}{n.nr} + \frac{2^2}{n.nr^2} + \frac{3^2}{n.nr^3}}$, &c. $=$ G------ But the firſt of theſe three ſerieſes is equal A, and the ſecond $= 2 \times \overline{A-V}$: Therefore the third, or $\frac{1^2}{n.nr} + \frac{2^2}{n.nr^2} + \frac{3^2}{n.nr^3}$, &c. $= G + 2 \times \overline{A-V} - A$; hence $n \times \overline{\frac{1^2}{n.nr} + \frac{2^2}{n.nr^2} + \frac{3^2}{n.nr^3}}$, &c. $= n \times \overline{G-A}$

$\times\,\overline{G-A} + 2n \times \overline{A-V}$, which being ſubtracted from $n \times \overline{A-V}$ found above we ſhall at laſt have $v = n \times \overline{V-G}$ - - - - Q. E. D.

This rule may be alſo very eaſily deduced from Dr. *Price*'s Theorem, given in his Treatiſe on Reverſionary Payments, page 302, 3d edition.—The ſame ſymbols being retained, let P denote the perpetuity, and p the value of 1 *l.* at the end of n years; then will Dr. *Price*'s Theorem for the whole continuance of life be expreſſed by $\overline{A-G} \times n + n.p.P - A.P.r = n \times \overline{A + Pp} - APr - nG$ - - - But $A + Pp$ is $= P$, and $P - \frac{APr}{n}$ is known to be $= V$. Therefore $n \times \overline{A+Pp} - APr = nV$; and $n \times \overline{A + Pp} - APr - nG = n \times \overline{V-G}$.

Note (E). PROBLEM XII. *Page* 128.

Let the value of the two *ſingle* lives be denoted by A and B; the value of the two joint lives by D; and the given ſum by S.—Then ſince the *whole* Annuity is to be enjoyed till the extinction of the joint lives, the value of an Annuity of 1 *l.* till this period will be expreſſed by D.

The value of one life after another is known to be equal to the *difference* between the *joint* lives and the life *in expectation.* Therefore the value of *half* the Annuity on the ſuppoſition that A will be the ſurvivor is $\frac{A-D}{2}$; and the ſame value on the ſuppoſition that B will be the ſurvivor, is $\frac{B-D}{2}$ - - - - The ſum of thoſe two values, or $\frac{A+B}{2}-D$, being added to D (the value of the whole Annuity during the joint lives) we have $\frac{A+B}{2}$ for the number of years purchaſe ſuch an Annuity is worth during the continuance of the joint lives, and the remainder of the ſurviving life. Hence as $\frac{A+B}{2}$ is to an Annuity of 1 *l.* (under the reſtrictions mentioned in the Problem) ſo is S to $\frac{2S}{A+B}$, which is the general rule.

Note (F). PROBLEM XXIII. *Page* 154.

CASE 1ſt. *Where B is ſuppoſed to be the oldeſt life.* —Let a, b and c, denote the complements of the lives of A, B and C, and let A be the youngeſt. The values of the 1ſt, 2d, 3d, &c. rents to b rents will be the probability, that all the three lives will have failed at the beginning of the 2d, 3d, 4th, &c. years to b years, (the life of c having failed firſt) multiplied reſpectively by $\frac{1}{r}$, $\frac{1}{r^2}$, $\frac{1}{r^3}$, &c. to $\frac{1}{r^b}$ (r denoting 1 *l.* increaſed by its intereſt for a year). —The probability that any ſingle life will have failed in any number of years is that number divided by the complement of life. The probabilities,

ties, therefore, that all the three lives whoſe complements are a, b and c will have failed in 1, 2, 3, &c. years (that is, at the beginning of the 2d, 3d, 4th, &c. years to b years) are $\frac{1^3}{abc}$, $\frac{2^3}{abc}$, $\frac{3^3}{abc}$, &c. to b terms. The probabilities that they ſhall have all failed at the beginning of the 2d, 3d, 4th, &c. years to b years, and that any one in particular of them (ſuppoſe the life whoſe complement is c) ſhall have failed firſt, are $\frac{1^3}{3\,abc}$, $\frac{2^3}{3\,abc}$, $\frac{3^3}{3\,abc}$, &c. to $\frac{b^3}{3\,abc}$ $\left(=\frac{b^2}{3\,ac}\right)$. The probability, therefore, that they ſhall have all failed (C having failed firſt) in b years, or at the beginning of the $\overline{b+1}$th year, is $\frac{b^2}{3\,ac}$; from whence it follows that the preſent value of the 1ſt, 2d, 3d, &c. rents to b rents is $\frac{1^3}{3\,abcr}+\frac{2^3}{3\,abcr^2}+\frac{3^3}{3\,abcr^3}$, &c. to $\frac{b^2}{3\,acr^b}=\frac{b^2}{3\,ac}\times\overline{\frac{1^3}{b^3r}+\frac{2^3}{b^3r^2}+\frac{3^3}{b^3r^3}}$, &c. to b terms; and that the value of the rents ever after, ariſing from the probability that all the lives ſhall die in b years (C having died firſt) is $\frac{b^2}{3\,acr^b}$ multiplied into $\overline{\frac{1}{r}+\frac{1}{r^2}+\frac{1}{r^3}}$, &c. *ad infinitum.* —But there is a further value of the ſeveral rents after b years, ariſing from the probability of A's dying *after* the expiration of this time, reſtrained however to the contingency of C's having been ſurvived by B. The probabilities that A dies after the expiration of b, $\overline{b+1}$, $\overline{b+2}$, &c. years are $\frac{1}{a}$, $\frac{2}{a}$, $\frac{3}{a}$, &c. The probability that C, the youngeſt of the two lives B and C, will die before B in b years is $\frac{b^2}{2bc}=\frac{b}{2c}$; which therefore multiplied into $\frac{1}{ar^{b+1}}$ will give the additional value of the $\overline{b+1}$th rent, and will be equal to $\frac{b}{2cr^b}\times\frac{1}{ar}$. In the ſame manner the additional values of the $\overline{b+2}$d, $\overline{b+3}$d, &c. rents will be $\frac{b}{2cr^b}\times\frac{2}{ar^2}$, $\frac{b}{2cr^b}\times\frac{3}{ar^3}$, &c. The ſum of all theſe values to the extremity of A's life, or to $\overline{a-b}$ terms, will be $=\frac{b}{2cr^b}\times\overline{\frac{1}{ar}+\frac{2}{ar^2}+\frac{3}{ar^3},\ \text{----}}$ $\overline{\text{\&c. to } \frac{a-b}{ar^a}}$. And the value of theſe rents ever after, ariſing from the probability of A's dying after B, and C's being ſurvived by B, is $\frac{b}{2cr^a}\times$ $\overline{\frac{a-b}{a}}\times\overline{\frac{1}{r}+\frac{1}{r^2}+\frac{1}{r^3}}$, *ad infinitum*, $=\frac{b}{2cr^a}\times$ $\overline{\frac{1}{r}+\frac{1}{r^2}+\frac{1}{r^3}}$, &c. $-\frac{bb}{2acr^a}\times\overline{\frac{1}{r}+\frac{1}{r^2}}$, &c. The whole preſent value, therefore, of all the rents is $\frac{b^2}{3ac}\times\overline{\frac{1^3}{b^3r}+\frac{2^3}{b^3r^2}+\frac{3^3}{b^3r^3}\text{------}(b)}+$

$\frac{b^2}{3acr^b} \times \overline{\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3}}$, &c. *ad infinitum*, $+ \frac{b}{2cr^b}$ $\times \overline{\frac{1}{ar} + \frac{2}{ar^2} + \frac{3}{ar^3} \cdots\cdots (a-b)} + \frac{b}{2cr^a} \times$ $\overline{\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3}}$, &c. *ad infinitum*, $- \frac{bb}{2acr^a} \times$ $\overline{\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3}}$, &c. *ad infinitum*. Let P = the value of the perpetuity, B = the value of the longeſt of three equal lives whoſe complement is b, A = the value of a ſingle life, whoſe complement is a, for $\overline{a-b}$ years. Then the foregoing expreſſion will be $= \frac{b^2}{ac} \times \overline{\frac{P-B}{3} - \frac{P}{2r^a}} + \frac{b}{2cr^b} \times$ $\overline{P-A}$, which is the general rule.

The ſolution of the ſecond Caſe is ſo very eaſy as hardly to require an explanation. For ſince the abſolute value of the Reverſion is compoſed of three parts, depending on the chance of B's dying before A and C, of A's dying before B and C, and C's dying before A and B, it is evident, that if the ſum of the two former be ſubtracted from the *whole* value, the remainder will give the number of years purchaſe required.

Note (G). PROBLEM XXVII. *Page* 166.

CASE FIRST. *When B is oldeſt, or when B is older than C, and C is older than A.* Let a, b, c and r denote the ſame quantities as in the preceding demonſtration. The values of the 1ſt, 2d, 3d, &c. rents to *b* rents may be conſidered as depending upon either of two events. Firſt, that A and C have *both* failed before the beginning of the 2d, 3d, &c. years, C having failed laſt, and B being alive at thoſe periods; ſecondly, that B and C are both alive and A dead at the beginning of the 2d, 3d, &c. years. The firſt of theſe probabilities is $\frac{\overline{b-1}.\ 1^2}{2abc}$ for the 1ſt year, $\frac{\overline{b-2}.\ 2^2}{2abc}$ for the 2d year, and ſo on to *b* years. The laſt of theſe probabilities for the 1ſt, 2d, 3d, &c. years will be $\frac{\overline{b-1}.\overline{c-1}.\ 1}{abc}$, $\frac{\overline{b-2}.\overline{c-2}.\ 2}{abc}$, &c. to *b* terms. Therefore the value of the 1ſt, 2d, 3d, &c. rents on the firſt contingency is $\frac{\overline{b-1}.\ 1^2}{2abcr} + \frac{\overline{b-2}\ 2^2}{2abcr^2}$ - - - - - - - (b) and the value of the ſame rents on the laſt contingency is $\frac{\overline{b-1}.\overline{c-1}.\ 1}{abcr} + \frac{\overline{b-2}.\overline{c-2}.\ 2}{abcr^2}$ + &c. - - - - - - - (b). Let B C = the value of the joint lives B and C, A B C = the value of the three joint lives, $\overset{|}{B}\ \overset{|}{C}$ = the value of the longeſt of the two lives B and C, and $\overset{|}{A}\ \overset{|}{B}\ \overset{|}{C}$ = the value of the longeſt of the three lives. Then will the laſt of the above ſerieſes be expreſſed by $\overline{BC - ABC}$, and the firſt of them by $\frac{\overset{|}{A}\overset{|}{B}\overset{|}{C} - \overset{|}{A}\overset{|}{C}}{2}$; the ſum of which is the general rule.

Case Second. *When B is younger than C, and C older than A.* By reaſoning as above, the value of the 1ſt, 2d, 3d, &c. rents to c rents will be

$$\frac{\overline{b-1}.\overline{c-1}.1}{abcr}+\frac{\overline{b-2}.\overline{c-2}.2}{abcr^2}+\&c.\text{ - - - }(c)$$

and $\frac{\overline{b-1}.1^2}{2abcr}+\frac{\overline{b-2}.2^2}{2abcr^2}$ - - - - - - (c). Now it is evident that B has a further chance of receiving the Annuity after this period depending upon the probability that he has ſurvived both A and C, A having died firſt. The value therefore of the $\overline{c+1}$th, $\overline{c+2}$d, &c. rents to the extremity of B's life will be this probability multiplied into $\frac{1}{r^{c+1}}$, $\frac{1}{r^{c+2}}$, &c. $(b-c)$ that is, it will be $=$

$$\frac{\overline{b-c-1}\times cc}{2abcr^{c+1}}+\frac{\overline{b-c-2}\times cc}{2abcr^{c+2}}+\&c.\text{ - - - - - - - }$$

$(b-c)$. Hence the whole preſent value of the Annuity will be $\frac{\overline{b-1}.\overline{c-1}.1}{abcr}+\frac{\overline{b-2}.\overline{c-2}.2}{abcr^2}$

$$\text{- - - - - }(c)+\frac{\overline{b-1}.1^2}{2abcr}+\frac{\overline{b-2}.2^2}{2abcr^2}+\text{ - - - - - -}$$

$$\frac{\overline{b-c}.c^2}{2abcr^c}+\frac{\overline{b-c-1}.c^2}{2abcr^{c+1}}+\text{ - - - - - - - - - }(b-c)$$

With regard to the laſt of theſe ſerieſes; ſince

$$\frac{1}{ccb}\times\frac{\overline{b-1}.1^2}{r}+\frac{\overline{b-2}.2^2}{r^2}\text{ - - - - }+\frac{\overline{b-c}.cc}{r^c}+$$

$\frac{\overline{b-c-1}.cc}{r^{c+1}}$ - - - - $(b-c)$ is the value of the Reverſion of the life of B after the longeſt of two equal lives C, C; it follows that the true value thereof muſt be equal to that of the ſaid Reverſion multiplied by $\frac{c}{2a}$. Let the ſame ſymbols be retained as in the preceding caſe, and let $\overset{\prime}{C}\,\overset{\prime}{C}$ = the value of the longeſt of two equal lives C, C, and $\overset{\prime}{B}\,\overset{\prime}{C}\,\overset{\prime}{C}$ = the value of the longeſt of three lives B, C and C; then will the whole value of the Annuity in this caſe be expreſſed by $\overline{BC - ABC} + \frac{c}{2a} \times \overline{\overset{\prime}{B}\,\overset{\prime}{C}\,\overset{\prime}{C} - \overset{\prime}{C}\,\overset{\prime}{C}}$.

CASE THIRD. *When B is younger than C, and C younger than A.* The truth of this rule is very obvious: For ſince the *abſolute* value of the Reverſion of the life of B after the joint lives of A and C is compoſed of two parts, depending on the chance of the younger dying before the elder, and on the chance of the elder dying before the younger; it follows, that if the value of the expectation on the *firſt* of theſe contingencies (given in the ſolution of Caſe 2d) be ſubtracted from the *abſolute* value of the Reverſion, that the remainder will be the value of the expectation on the *latter* of them.

Note (H). PROBLEM XXIX. *Page* 175.

CASE FIRST. *When C is oldeſt.* Let *a*, *b*, *c* and *r* ſtill denote the ſame quantities: Then ſince it appears from the demonſtration of Problem 23d that $\frac{1^3}{3abc}$, $\frac{2^3}{3abc}$, &c. repreſent the probabilities that the three lives ſhall have failed at the beginning of the 2d, 3d, &c. years, and that one of them in particular (ſuppoſe the life whoſe complement is *a*) ſhall have failed firſt; it is evident that if the foregoing fractions are ſeverally diminiſhed one half, they will expreſs the probability of ſurvivorſhip required in this Problem; that is, $\frac{1^3}{3abc} \times \frac{1}{2}$, $\frac{2^3}{3abc} \times \frac{1}{2}$, &c. will denote the probabilities that the three lives are dead at the beginning of the 2d, 3d, &c. years, and that A has died firſt, and either of the other two (C for inſtance) has died laſt. For as it is an equal chance which of the two ſurviving lives B and C ſhall die firſt, it is plain that if the former probabilities *without* this reſtriction are leſſened in the proportion of $\frac{1}{2}$, we ſhall have the probabilities of ſurvivorſhip *with* this reſtriction.—The value, therefore, of the rents during the 1ſt, 2d, 3d, &c. years to *c* years will be

$$\overline{\frac{1^3}{6abcr} + \frac{2^3}{6abcr^2} + \text{\&c.} \text{-----} (c)} = \frac{c^2}{6ab} \times \overline{\frac{1^3}{c^3 r} + \frac{2^3}{c^3 r^2} + \text{-----} \frac{c^3}{c^3 r^c}},$$

and the values of all the rents ever after will be $\frac{c^2}{6abr^c} \times \overline{\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3}}$ *ad infinitum*; for $\frac{c^3}{6abc}$ expreſſes the probability of ſurvivorſhip during the whole continuance of C's life, at the extinction of which A and B muſt be dead (B having died laſt) to entitle to the receipt of the following rents; that is, to the rents after c years. Now thoſe rents being ſimply $= \frac{1}{r^{c+1}} + \frac{1}{r^{c+2}} + \frac{1}{r^{c+3}}$ &c. *ad infinitum*, if they are diminiſhed in the proportion of the probability of this ſurvivorſhip, or multiplied by $\frac{c^2}{6\,ab}$, we ſhall have their true value. But $\frac{c^2}{6\,ab} \times \overline{\frac{1}{r^{c+1}} + \frac{1}{r^{c+2}} \text{ \&c.}}$ is the value of the perpetuity after c years multiplied into $\frac{c^2}{6\,ab}$, and the firſt ſeries is equal to the difference between an Annuity certain for c years and the value of the longeſt of three equal lives, whoſe complement is c, multiplied into the ſame co-efficient: Hence if $P =$ the perpetuity, and $C =$ the value of the three lives juſt mentioned, the two ſerieſes $\frac{c^2}{6ab} \times \overline{\frac{1^3}{c^3 r} + \frac{2^3}{c^3 r^2} + \cdots\cdots \frac{c^3}{c^3 r^c}}$, and $\frac{c^2}{6abr^c} \times \overline{\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3}}$ &c. *ad infinitum*, expreſſing the whole preſent value of the Reverſion, will become equal to $\frac{c^2}{6ab} \times \overline{P - C}$, which is the general rule.

Case Second. *When B is oldeſt.* By reaſoning as in the preceding Caſe, the value of the rents for the 1ſt, 2d, 3d, &c. years to *b* years are $\frac{1^3}{6abcr}$, $\frac{2^3}{6abcr^2}$, $\frac{3^3}{6abcr^3}$, &c. to $\frac{b^3}{6abcr^b} = \frac{b^2}{6ac} \times \overline{\frac{1^3}{br^3} + \frac{2^3}{b^3 r^2}, \text{\&c. to } \frac{b^3}{b^3 r^b}}$, and the value of the rents ever after, that is of the $\overline{b+1}$th, $\overline{b+2}$d, &c. rents, *ad infinitum*, ariſing from the probability that all the three lives ſhall die in *b* years, A having died firſt and C having died laſt, is $\frac{b^2}{6ac} \times \overline{\frac{1}{r^{b+1}} + \frac{1}{r^{b+2}} + \frac{1}{r^{b+3}}, \text{\&c.}} = \frac{b^2}{6acr^b} \times \overline{\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3}}$, &c. *ad infinitum.* But there is a further expectation beyond this period (or the limit of B's life) depending on the probability of C's dying *after* B and A's dying *before* him. The firſt of theſe probabilities for the 1ſt, 2d, 3d, &c. years to $\overline{c-b}$ years, is $\frac{1}{c}$, $\frac{2}{c}$, $\frac{3}{c}$, &c. to $\overline{c-b}$ terms. The laſt is conſtantly expreſſed by $\frac{b}{2a}$. Hence the additional value of the rents after the limit of B's life for the remainder of C's exiſtence, will be for the $\overline{b+1}$th year $\frac{1}{cr^{b+1}} \times \frac{b}{2a}$, for the $\overline{b+2}$d year $\frac{2}{cr^{b+2}} \times \frac{b}{2a}$, and ſo on to the $\overline{c-b}$th year, the value of which rent will be $\frac{\overline{c-b} \times b}{2car^{c}}$. Theſe ſeveral values being added together, will be equal $\frac{b}{2ar^{b}} \times \overline{\frac{1}{cr} + \frac{2}{cr^{2}} + \frac{3}{cr^{3}} \cdots\cdots + \frac{\overline{c-b}}{cr^{c-b}}}$. Again, the values of the rents for ever after the limit of C's life, ariſing from the probability that C ſhall have died *after* B, and that B ſhall have ſurvived A, will be for the $\overline{c+1}$th rent $\frac{\overline{c-b}.\, b}{2ca} \times \frac{1}{r^{c+1}}$, for the $\overline{c+2}$d, $\overline{c+3}$d, &c. rents, *ad infinitum*, $\frac{\overline{c-b}.\, b}{2ca} \times \frac{1}{r^{c+2}}$, $\frac{\overline{c-b}.\, b}{2ca} \times \frac{1}{r^{c+3}}$, &c. $= \frac{\overline{c-b}.\, b}{2acr^{c}} \times \overline{\frac{1}{r} + \frac{1}{r^{2}} + \frac{1}{r^{3}}, \&c.} = \frac{b}{2ar^{c}} \times \overline{\frac{1}{r} + \frac{1}{r^{2}} + \frac{1}{r^{3}}, \&c.} - \frac{bb}{2acr^{c}} \times \overline{\frac{1}{r} + \frac{1}{r^{2}} + \frac{1}{r^{3}}, \&c.}$ The whole preſent value, therefore, of all the rents is $\frac{b^{2}}{6ac} \times \overline{\frac{1^{3}}{b^{3}r} + \frac{2^{3}}{b^{3}r^{2}} + \frac{3^{3}}{b^{3}r^{3}} \cdots (b)} + \frac{b^{2}}{6acr^{b}} \times \overline{\frac{1}{r} + \frac{1}{r^{2}} + \frac{1}{r^{3}}, \&c.} - \frac{bb}{2acr^{c}} \times \overline{\frac{1}{r} + \frac{1}{r^{2}} + \frac{1}{r^{3}}, \&c.} + \frac{b}{2ar^{b}} \times \overline{\frac{1}{cr} + \frac{2}{cr^{2}} + \frac{3}{cr^{3}} \cdots\cdots \overline{c-b}} + \frac{b}{2ar^{c}} \times \overline{\frac{1}{r} + \frac{1}{r^{2}}, \&c.}$ Let P = the value of the perpetuity, B = the value of the longeſt of three equal lives whoſe complement is b, and $\beta =$ the value of an Annuity on the life of C for $\overline{c-b}$ years, then the foregoing expreſſion will be $\frac{b^2}{ac} \times \overline{\frac{\overline{P-B}}{6} - \frac{P}{2r^c}} + \frac{b}{ar^b} \times \frac{\overline{P-\beta}}{2}$ - - - - which is the rule.

Case Third. *When A is oldeſt and B youngeſt.* By proceeding as in the two former Caſes, it will appear that the value of all the rents during the limit of A's life, that is for the 1ſt, 2d, 3d, &c. years to a years, will be $\frac{a^2}{6bc} \times \overline{\frac{1^3}{a^3r} + \frac{2^3}{a^3r^2} + \frac{3^3}{a^3r^3}}$ - - - - (a), and that the value of all the rents ever after, ariſing from the probability that the three lives ſhall die in a years, B having died 2d, and C having died laſt, will be $\frac{a^2}{6bcr^a} \times \overline{\frac{1}{r} + \frac{1}{r^2}}$ &c. But this laſt ſeries by no means expreſſes the whole value of the $\overline{a+1}$th, $\overline{a+2}$d, &c. rents. There are two other contingencies, upon the happening of either of which the expectant may come into poſſeſſion of the eſtate. Firſt, that B and C are living at the end of a years, and that they ſhall *both* be dead at the beginning of the $\overline{a+2}$d, $\overline{a+3}$d, &c. years, (C having previouſly ſurvived B); the probabilities of which contingency are ſeverally expreſſed by $\frac{1^2}{2bc}$, $\frac{2^2}{2bc}$, $\frac{3^2}{2bc}$, &c. The value, therefore, of the $\overline{a+1}$th rent ariſing from the chance of this event's happening is $\frac{1^2}{2bcr^{a+1}}$; the values of the $\overline{a+2}$d, $\overline{a+3}$d, &c. rents to $\overline{c-a}$ rents will be $\frac{2^2}{2bcr^{a+2}}$, $\frac{3^2}{2bcr^{a+3}}$, &c. to $\frac{\overline{c-a}|^2}{2bcr^c}$. The ſum of all which is equal to $\frac{c}{2br^a} \times \overline{\frac{1^2}{c^2r} +}$ $\overline{\frac{2^2}{c^2r^2} + \frac{3^2}{c^2r^3} \text{- - - - - - - - -} + \frac{\overline{c-a}|^2}{cc.r^{c-a}}}$. And the value of all the rents ever after, ariſing from the probability that B and C are both dead at the end of c years, that C has died laſt, and that both of them have ſurvived the a firſt years, is $\frac{\overline{c-a}|^2}{2bcr^c} \times$ $\overline{\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3}}$, *ad infinitum.* The ſecond contingency, which may bring the expectant into poſſeſſion of the eſtate, after the limit of A's life, ariſes from the probability of B's dying during the ſaid limit (having previouſly ſurvived A) and C's dying *after* this limit. The probability that B dies in a years after ſurviving A, is conſtantly expreſſed by $\frac{a}{2b}$. The probability that C is dead at the beginning of the $\overline{a+1}$th year is $\frac{1}{c}$, at the beginning of the $\overline{a+2}$d year $\frac{2}{c}$, and ſo on to the $\overline{c-a}$th year. The values therefore of the $\overline{a+1}$th, $\overline{a+2}$d, &c. rents to $\overline{c-a}$ rents ariſing from the probability of this contingency are $\frac{a}{2b} \times \frac{1}{cr^{a+1}}$,

$\frac{a}{2b} \times \frac{2}{cr^{a+2}}$, &c. to $\frac{a}{2b} \times \frac{c-a}{cr^c} = \frac{a}{2br^a} \times \overline{\frac{1}{cr} + \frac{2}{cr^2} + \frac{3}{cr^3} \text{------} + \frac{c-a}{cr^{c-a}}}$. And the values of the rents ever after, that is, the values of the $\overline{c+1}$th, $\overline{c+2}$d, $\overline{c+3}$d, &c. rents, *ad infinitum*, ariſing from the probability that C has ſurvived the firſt a years, and that B has died during this time *after* A, are $\frac{a}{2b} \times \frac{\overline{c-a}}{cr^c} \times \frac{1}{r}$, $\frac{a}{2b} \times \frac{\overline{c-a}}{cr^c} \times \frac{1}{r^2}$, &c. $= \frac{a}{2b} \times \frac{\overline{c-a}}{cr^c} \times \overline{\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3}}$ &c. *ad infinitum*. The whole preſent value, therefore, of all the rents is $\frac{a^2}{6bc} \times \overline{\frac{1^3}{a^3r} + \frac{2^3}{a^3r^2} + \frac{3^3}{a^3r^3} \text{-----} (a)} + \frac{a^2}{6bcr^a} \times \overline{\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3}}$ &c. *ad infinitum*, $+ \frac{c}{2br^a} \times \overline{\frac{1^2}{c^2r} + \frac{2^2}{c^2r^2} + \frac{3^2}{c^2r^3} \text{ \&c. ------} (c-a)} + \left(\frac{\overline{c-a}|^2}{2bcr^c} =\right) \frac{c^2 - 2ac + a^2}{2bcr^c} \times \overline{\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3}}$ *ad infinitum*, $+ \frac{a}{2br^a} \times \overline{\frac{1}{cr} + \frac{2}{cr^2} + \frac{3}{cr^3} \text{-----} (c-a)} + \frac{ac - a^2}{2bcr^c} \times \overline{\frac{1}{r} + \frac{1}{r^2}}$ &c. $= \frac{a^2}{6bc} \times \overline{\frac{1^3}{a^3r} + \frac{2^3}{a^3r^2} + \frac{3^3}{a^3r^3} \text{-------} (a)} + \frac{a^2}{6bcr^a} \times \overline{\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3}}$ &c. *ad in-*

$$\textit{finitum}, + \frac{c}{2br^a} \times \overline{\frac{1^2}{c^2 r} + \frac{2^2}{c^2 r^2} \cdots\cdots \overline{(c-a)}} +$$

$$\frac{c}{br^c} \times \overline{\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3}}\ \&c. + \frac{a}{2br^a} \times \overline{\frac{1}{cr} +}$$

$$\overline{\frac{2}{cr^2} + \frac{3}{cr^3} \cdots \overline{(c-a)}} + \frac{a}{2br^c} \times \overline{\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3}}$$

$$\&c.\ \textit{ad infinitum}, - \frac{a}{br^c} \times \overline{\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3}}\ \&c.$$

ad infinitum - - - - - Let P = the value of the perpetuity, A = the value of the longeſt of three equal lives whoſe complement is a, K = the value of the longeſt of two equal lives, whoſe complement is c, for $\overline{c-a}$ years, and x = the value of a ſingle life of the ſame age for the ſame term; then the foregoing expreſſion will be =

$$\frac{a^2}{6bc} \times \overline{P-A} + \frac{c.\ \overline{P-K} + a.\ \overline{P-x}}{2br^a} -$$

$\frac{aP}{br^c}$, which is the general rule.

Caſe Fourth. *When A is oldeſt and C youngeſt.* By reaſoning exactly as in Caſe 3d, the value of the rents during the limit of A and B's lives, and ever after, ariſing from the probability of all the three lives being dead at the end of b years, (A having died firſt and C having died laſt) will be found equal to $\frac{a^2}{6bc} \times \overline{P-A} + \frac{b.\overline{P-B} + a.\overline{P-\beta}}{2cr^a}$ $- \frac{aP}{cr^b}$ (β and B repreſenting the values of Annuities on the ſingle and the longeſt of two equal lives, whoſe complement is b, for $\overline{b - a}$ years, and P and A repreſenting the ſame values as in Caſe 3d).——Now it is evident that the above does not expreſs the *whole* value of the eſtate: For the rents after b years; that is, after the neceſſary extinction of A and B's lives, have an additional value from the probability of C's ſurviving this period, and being dead at the beginning of the $\overline{b + 2}$d, $b + 3$d, &c. years to $\overline{c - b}$ years. Theſe probabilities are ſeverally expreſſed by $\frac{1}{c}$, $\frac{2}{c}$, $\frac{3}{c}$, &c. - - - - to $\frac{c - b}{c}$. Therefore the additional value of the $\overline{b + 1}$th, $\overline{b + 2}$d, $\overline{b + 3}$d, &c. rents to the $\overline{c - b}$th rent, will be $\frac{1}{cr^{b+1}} + \frac{2}{cr^{b+2}} + \frac{3}{cr^{b+3}}$ - - - - - to $\frac{c - b}{cr^{c}} =$ $\frac{1}{r^{b}} \times \overline{\frac{1}{cr} + \frac{2}{cr^{2}} + \frac{3}{cr^{3}} \text{ - - - - - } \frac{c - b}{cr^{c-b}}}$. And the value ever after, ariſing from the probability of C's having ſurvived b years, will be $\frac{c - b}{cr^{c}} \times \frac{1}{r} +$ $\frac{c - b}{cr^{c}} \times \frac{1}{r^{2}}$ &c. *ad infinitum*, $= \frac{c - b}{cr^{c}} \times \overline{\frac{1}{r} +}$ $\overline{\frac{1}{r^{2}} + \frac{1}{r^{3}}}$ &c. *ad infinitum*. Hence the whole additional value of the rents after b years will be $=$ $\frac{1}{rb} \times \overline{\frac{1}{cr} + \frac{2}{cr^{2}} + \frac{3}{cr^{3}} \text{ - - - - - - } (c - b)} + \frac{1}{r^{c}} \times$ $\overline{\frac{1}{r} + \frac{1}{r^{2}} + \frac{1}{r^{3}}}$ &c. $- \frac{b}{cr^{c}} \times \overline{\frac{1}{r} + \frac{1}{r^{2}}}$ &c. $=$

$\frac{\overline{P-C}}{r^c} - \frac{bP}{cr^c}$ (C repreſenting the value of an Annuity on a life, whoſe complement is c, for $\overline{c-b}$ years). This expreſſion, therefore, added to the above, will be the whole value of all the rents, and is equal to $\frac{a^2}{6bc} \times \overline{P-A} + \frac{b.\overline{P-B} + a.\overline{P-\beta}}{2cr^a}$ $- \frac{aP}{cr_b} + \frac{P-C}{rc} - \frac{bP}{cr^c}$, which is the general rule.

Mr. *Simpſon* in his Select Exercifes (page 328) has given the fluent $\frac{Sq^x}{2abcm} \times \overline{x^2 - \frac{2x}{m} + \frac{2}{m}}$ (when properly corrected) as the true value of the ſum S to be received on theſe contingencies; q denoting unity divided by the rate of intereſt, m the hyperbolic logarithm of q, and x the interval of time during which the value is required. But this fluent by no means expreſſes the whole value, except in the ſingle caſe where C is the oldeſt of the three lives.——By a *fluxional* calculus of the ſame kind with Mr. *Simpſon*'s, the following Theorems may be obtained, which expreſs correctly the whole value in every caſe.

THEOREM

THEOR. 1ſt.

When C is the oldeſt.

$= \frac{S}{2\,abcr} \times \overline{\overline{A - pc} \times 2P^2 - Ppc^2}$ ----- A being the value of an Annuity certain for the time c, p the value of 1 *l.* at the end of c years, r the value of 1 *l.* increaſed by its intereſt for a year, P the perpetuity, S the given ſum, and a, b, c the reſpective complements of A, B, and C.

THEOR. 2d.

When B is oldeſt.

$= \frac{S}{2\,abcr} \times \overline{\overline{B - \pi b} \times 2P^2 - P\pi b^2} + \frac{Sb}{2\,ac} \times \frac{N}{r^{b+1}}$ ----- S, a, b, c, r, and P being as in Theorem 1ſt, B the value of an Annuity certain for b years, π the value of 1 *l.* at the end of b years, and N the value of an Annuity certain for $\overline{c - b}$ years.

THEOR. 3d.

When A is oldeſt and B youngeſt.

$= \frac{S}{2\,abcr} \times \overline{\overline{C - \kappa a} \times 2P^2 - P\kappa a^2} + \frac{S}{bc} \times \frac{\overline{\overline{A - \alpha} \times \overline{c - a}}}{r^{a+1}} + \frac{Aa}{2r^{a+1}}$ ---- S, a, b, c, r, and P being as above, C the value of an Annuity certain for (a) years, κ the value of 1 *l.* at the end of (a) years, A the value of an Annuity certain for $\overline{c - a}$ years, and α the value of an Annuity on the life whoſe complement is $\overline{c - a}$.

Theor. 4th.

When A is oldeſt and C youngeſt.

$$= \frac{S}{2abcr} \times \overline{\overline{C - xa} \times 2P^2 - Pxa^2}$$

$$+ \frac{S}{c} \times \overline{\frac{A - \beta}{r^{a+1}} + \frac{a \times 2\overline{\beta - B}}{2br^{a+1}}} \cdots$$

S, a, b, c, r, P, A and x repreſenting the ſame quantities as in Theorem 3d, B the value of an Annuity certain for $\overline{b - a}$ years, and β the value of a life whoſe complement is $\overline{b - a}$.

It would take up too much time to go through the ſeparate inveſtigations of thoſe Theorems; though perhaps it may not be proper intirely to omit them. I ſhall, therefore, give the inveſtigation of the 4th Theorem, which comprehends in it the chief reaſoning that has been applied to the ſolution of the three foregoing ones.

Investigation of Theorem 4th.—By proceeding in the ſame manner with Mr. *Simpſon* and retaining his ſymbols, the expectation during the time x is defined by the fluent $\frac{Sq^x}{2abcrm} \times \overline{x^2 - \frac{2x}{m} + \frac{2}{m^2}}$, which being corrected (by firſt ſuppoſing x to vaniſh, and then to become equal to (a),) is changed into $\frac{aSq^a}{2bcmr} - \frac{Sq^a}{bcm^2r} + \frac{S}{abcm^3r} \times \overline{q^a - 1}$, which gives the whole value of the expectation for the firſt (a) years.—Let P be now ſubſtituted for its equal $-\frac{1}{m}$, C for $\overline{1 - q^a} \times -$

$\frac{1}{m}$, and x for q^a, then will the above expreſſion be at laſt reduced to $\frac{S}{2abcr} \times \overline{\overline{C - xa} \times 2P^2 - Pxa^2}$.

The value of the expectation after the time (a) depends on the contingency of C's dying *after* the greateſt limit of A's life, having previouſly ſurvived B.—Now as B may have died *within* the limit of A's life, though *after* the deceaſe of A, this probability muſt be taken into conſideration; which (by reaſoning as Mr. *Simpſon* has done page 316 of his Select Exerciſes) will be expreſſed by $\frac{a}{2b}$. The probability of B's dying in x time after this period is $\frac{x}{b}$. Theſe two fractions added together give $\frac{a + 2x}{2b}$ for the *whole* probability of B's dying in $\overline{a + x}$ time according to the order in the Problem. The value, therefore, of the expectation on the death of C, during the moment $\dot{x}$ and under theſe circumſtances, will be properly repreſented by $\frac{a + 2x}{2b} \times \frac{\dot{x}}{c} \times Sq^x = \frac{aS}{2bc} \times q^x\dot{x} + \frac{S}{bc} \times q^x x\dot{x}$ - - - - The fluent of the firſt expreſſion generated while x is increaſing from a to b will be $\frac{aS}{2bc} \times$ B*. The 2d expreſſion may be changed into $\frac{b-a.\,S}{bc} \times \frac{q_x x\dot{x}}{b-a}$, whoſe fluent generated as above is equal to $\frac{\overline{b-a}.\,S}{bc} \times \overline{B-\beta}$*. As this Reverſion, expreſſed by both the preceding quantities, is not to take place till (*a*) years hence, and from the nature of the computation requires to be corrected by Dr. *Price*'s rule†, it muſt be diſcounted for $\overline{a+1}$ years, and conſequently will be reduced to $\frac{aS \times B}{2\,bcr^{a+1}} + \frac{\overline{b-a}.\,S.}{bc} \times \frac{\overline{B-\beta}}{r^{a+1}}$ - - - - -. The third and laſt value of the expectation ariſes from the chance of C's not dying till after the extinction of both A and B. - - - - - This expectation during the moment $\dot{x}$ will be $Sq^x \times \frac{\dot{x}}{c}$, whoſe fluent, generated while x is increaſing from b to c, will be the value of the Reverſion, and is equal to $\frac{S}{c}$ drawn into M, the value of an Annuity certain for $\overline{c-b}$ years‡. This being diſcounted for b years and corrected by Dr. *Price*'s Rule, as in the caſe of the preceding expectation, will be expreſſed by $\frac{S}{c} \times \frac{M}{r^{b+1}}$. Hence the *whole* value of all the expectations is $\frac{S}{2\,abcr} \times \overline{\overline{C-xa} \times 2P^2 - Pxa^2} +$

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* See *Simpſon*'s Select Exerciſes, page 324.

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* See *Simpſon*'s Select Exerciſes, page 327.

† See Treatiſe on Reverſionary Payments, Scholium to Queſtion 10th.

‡ See *Simpſon*'s Select Exerciſes, page 324.

$\frac{aS}{2bc} \times \frac{B}{r^{a+1}} + \frac{\overline{b-a}.S}{bc} \times \frac{\overline{B-\beta}}{r^{a+1}} + \frac{S}{c} \times \frac{M}{r^{b+1}}$; which, by ſubſtituting $\frac{A}{r^{a+1}}$ for its equal $\overline{\frac{B}{r^{a+1}} + \frac{M}{r^{b+1}}}$, &c. is at laſt reduced to $\frac{S}{2abcr} \times \overline{\overline{C - xa \times 2P^2 - Pxa^2}} + \frac{S}{c} \times \overline{\frac{A-\beta}{r^{a+1}} + \frac{a.\overline{2\beta-B}}{2br^{a+1}}}$ . . . :

Q. E. D.

The values by thoſe Theorems in moſt caſes agree nearly with the ſame values found by the Rules in page 175, &c. 1ſt. Let the given ſum be 1000 *l.* the ages as in the ſeveral examples of Problem 29th, and let C be the *oldeſt* life. Then by the firſt Theorem the value of the Reverſion will be equal to 10.808 *l.* which by the algebraical ſolution is determined to be 14.68 *l.*—2dly, Let B be the oldeſt life. Then by proceeding according to Theorem 2d, the reverſionary value will be found equal to 31.913 *l.* which by the algebraical ſolution is 36.035 *l.*—3dly, Let A be the oldeſt and C the youngeſt life. The value of the Reverſion by Theorem 3d is = 52.411 *l.* which by the algebraical ſolution is = 56.577 *l.*——Laſtly, Let A be the oldeſt and B the youngeſt. The reverſionary value by Theorem 4th is = 65.56 *l.* and the ſame value by the algebraical ſolution is = 67.881 *l.*

It may be obſerved that the values of the Reverſions by the fluxionary ſolutions are conſiderably *leſs* than thoſe by the algebraical ones. The reaſon is, that in computing from the former as well as the latter, the values of the Life-Annuities and Annuities certain have been taken on the ſuppoſition that they are payable *yearly*, whereas the fluxional ſolution ſuppoſes that they are payable *momently*.—Had the values of Annuities payable *momently* been taken, the reſults from both methods of ſolution would have been the ſame.—But if the values of Annuities payable *momently* are taken in computing from the fluxional ſolution, and of Annuities payable *yearly* in computing from the algebraical ſolution, the reſults will be always *leſs* from the former than from the latter; becauſe the values of Annuities payable *momently* being greater than the values of Annuities payable *yearly*, the values of eſtates or ſums of money to be received after their extinction, muſt be proportionably leſs. —All this is true, ſuppoſing the values of Annuities payable momently or yearly are given *accurately*. But if an approximation is uſed (as hath been done in the preceding examples in finding by Problem 6th the values of Annuities on the longeſt of three lives) another cauſe of difference will ariſe, which will be more or leſs as the approximation comes more or leſs near to exactneſs. —In order to illuſtrate all this, I ſhall beg leave to give the following examples, where the life and certain Annuities are ſuppoſed to be payable *momently*, and the values of Annuities on the longeſt of three lives are determined *exactly*.

EXAMPLE FIRST. Let the age of A be 20, that of B 30, and that of C 60 years. Let the ſum be 100 *l.* and, for greater perſpicuity, let the decrements of life alſo be according to *De Moivre's* hypotheſis.—This Caſe belongs to the firſt fluxional Theorem, where a, b, and c are reſpectively equal to 66, 56, and 26; hence $\frac{S}{2\,abc} = \frac{100}{192192}$ - - - - - A may be found = 16.167 - - - $p = .353$; therefore, P being equal 25, $\overline{A - pc} \times 2P^2$ is = 8736.25; and Ppc^2 is = 5965.7. Conſequently $\frac{S}{2\,abcr} \times \overline{\overline{A - pc} \times 2P^2 - Ppc^2} = \frac{100}{192192} \times \frac{8736.25 - 5965.7}{1.04} = 1.38\,l$, which is the value required.

Now let the ſame value be inveſtigated by Rule 1ſt, Problem 29th. - - - An Annuity on the longeſt of three equal lives aged 60 may be found equal to 13.16, which ſubtracted from 25 (the perpetuity) and the remainder being divided by 6, becomes = 1.9733. Therefore as (56 × 66 =) 3696 is to (26 × 26 =) 676 ſo is 1.9733 to .3609; which multiplied into 100, and the product divided by 26, we have 1.38 *l.* for the anſwer as above.

Example Second. Let the age of A be 20, that of B 60, and that of C 30 years. The ſum 100 *l.*——This Caſe belongs to the 2d fluxional Theorem, where, by proceeding as in the 1ſt example, $\frac{S}{2\,abcr} \times \overline{\overline{B - \pi b} \times 2P^2 - P\pi b^2}$ will be found equal to 1.38. N (the value of an Annuity certain for 30 years) is = 17.47; and $\frac{N}{r^{b+1}}$ = 17.47 × .34 = 5.9398. Therefore $\frac{bS}{2ac} \times \frac{N}{r^{b+1}}$ is = $\frac{2600}{7392} \times 5.9398 = 2.09$, which added to 1.38, found above, gives 3.47 *l.* for the value required.

Let this value be now ſought by Rule 2d, Problem 29th.—By following the ſame ſteps as in the preceding example the $\frac{1}{6}$th remainder will be found = 1.9733. 12.5 (half the perpetuity) multiplied into .106 (the value of 1 *l.* at the end of 56 years) is = 1.325 - - - - - As, therefore, 3696 is to 676 ſo is (1.9733 − 1.325 =) .648 to .1186; which muſt be *reſerved.* - - - - - The value of an Annuity on the life of C for 30 years is = 13.712; which ſubtracted from 25 (the perpetuity) and divided by 2 quotes 5.644. Hence as 33 is to 13 ſo is 5.644 to 2.223, which multiplied into .353 (the value of 1 *l.* at the end of 26 years) is reduced to .7847, and this added to .1186 (*reſerved* above) gives .9033, which being again multiplied into 100 and then divided by 26, we have 3.47 *l.* for the value required, as in the fluxional ſolution.

By proceeding in the ſame manner, the two other Theorems will be found to agree exactly with the correſponding Rules in Problem 29th. It will take up too much time to go through theſe operations; but the values reſulting from them, as well as thoſe inveſtigated above, and alſo the ſame values computed from different rates of mortality appear, when compared with each other, to be as follow.

EXACT *Values of* 100 *l. ſuppoſing Annuities payable* MOMENTLY*.

	By the fluxional Rule and *De Moivre*'s Hypotheſis.	By the algebraical Rule and *De Moivre*'s Hypotheſis.
Caſe 1ſt,	1.38	1.38
Caſe 2d,	3.47	3.47
Caſe 3d,	5.41	5.41
Caſe 4th,	6.68	6.68

Values of 100 *l. ſuppoſing Annuities payable* YEARLY.

	By the fluxional Rule and the Hypotheſis.	By the algebraical Rule, the Hypotheſis and the Approximation in Problem 6th.	By the algebraical Rule, the *Northampton* Table, and the Approximations in Problems 2d, 3d, and 6th.	By the algebraical Rule, the *Northampton* Table, and taking the *exact* Values of the Lives.
Caſe 1ſt,	1.08	1.40	1.381	1.384
Caſe 2d,	3.19	3.60	3.531	3.524
Caſe 3d,	5.24	5.65	5.513	5 429
Caſe 4th,	6.55	6.78	6.753	6.680

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* The Value of an Annuity on the *longeſt* of three lives has been found by the Rule in Problem 2, Corollary 2, *Simpſon's Annuities.* The values of the *joint* lives (neceſſary for determining thoſe on the longeſt of three lives) have alſo been found by a fluxional Rule given by the ſame author in his *Select Exercifes*, page 325.—And the values of the *ſingle* life, and certain Annuities have been found by Dr. *Price*'s Theorems in the Philoſophical Tranſactions, Vol. 66, No. 6.

Values of 100 *l. ſuppoſing Annuities payable* momently *in computing by the fluxional Rule, and that in computing by the algebraical Rule they are payable* yearly *and the Approximation in Problem 6th is uſed.*

	By the fluxional Rule and *De Moivre's* Hypotheſis.	By the algebraical Rule and *De Moivre's* Hypotheſis.
Caſe 1ſt	1.38	1.40
Caſe 2d,	3.47	3.60
Caſe 3d,	5.41	5.65
Caſe 4th,	6.68	6.78

Theſe values are ſet down in this order to give a clearer idea of what has been ſaid in the preceding pages.—I ſhall only obſerve further on this ſubject, that the *algebraical* Rules are generally, if not always, preferable, as being the moſt ſimple; and that the values of the Life and certain Annuities may be taken as payable *yearly* without any material error, even though the Reverſions are in *landed eſtates*, which are the only caſes where the *fluxional* ſolutions ought to be applied.

Note (I). PROBLEMS XXX, XXXI, &c. *Pag.* 185, &c.

PROBLEM XXX. Let L, *n*, and *l* ſignify the numbers of the living in the Table at the ages of A, B, and C - - - - - D′, D″, D‴, &c. the decrements of life in the Table at the end of 1, 2, 3,

&c. years from the age of A - - - - Δ', Δ'', Δ''', &c, and d', d'', d''', &c. the decrements from the ages of B and C - - - - e, f, g, &c. and o, p, q, &c. the numbers of the living in the Table after 1, 2, 3, &c. years from the ages of C and B reſpectively. - - - - Let r alſo denote 1 *l.* increaſed by its intereſt for a year, S the given ſum, and t the given number of years. - - - Then by reaſoning as Mr. *Simpſon* has done in pag. 321, &c. of his *Select Exerciſes*, the value of B's expectation will be expreſſed by $\frac{S\times o}{n}\times\frac{D'\times d'}{2Llr}+\frac{S\times p}{n}\times\overline{\frac{d'\times D''}{Llr^2}+\frac{d''\times D''}{2Llr^2}}+\frac{S\times q}{n}\times\overline{\frac{\overline{d'+d''}.D'''}{Llr^3}+\frac{d''\times D'''}{2Llr^3}}$ &c. $(t)=\frac{S}{L}\times\overline{\frac{o.D'}{nr}+\frac{p.D''}{nr^2}+\frac{q.D'''}{nr^3}\ \&c.}$ $\overline{- - - - (t)}-\frac{S}{L}\times\overline{\frac{\overline{l-d'}\times oD'}{nlr}+\frac{\overline{l-\overline{d'+d''}}\times pD''}{nlr^2}+\frac{\overline{l-\overline{d'+d''+d'''}}\times qD'''}{nlr^3}}$, &c. - - - - $(t)-\frac{S}{2Lln}\times\overline{\frac{o.d'.D'}{r}+\frac{p.d''.D''}{r^2}+\frac{q.d'''.D'''}{r^3}}$, &c. (t) which is equal nearly to $\frac{S}{a}\times\overline{N-M}-\frac{S}{2ac}\times N$*, - - - - M repreſenting the value of the joint lives of B and C for *t* years, N the value of the life of B for the ſame term, and *a* and *c* the *complements* of the lives of A and C for *t* years *.

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* The laſt term in this Theorem, or $\frac{S}{2ac}\times N$, is of no conſequence when the condition of Survivorſhip is extended

This Theorem will give anſwers ſufficiently correct if the term is not leſs than five or ſix years, and the values of the ſingle and joint lives can be obtained true to three places of decimals. But if, for want of proper tables, ſuch values cannot be procured, and the term is ſhort, it will be neceſſary and it will alſo be eaſieſt to find the anſwer from the ſeries itſelf rather than the Theorem. For inſtance, were the term *three years* in the example to this Problem inſtead of 16 years, the firſt ſeries in the preceding inveſtigation taken as far as I have carried it will expreſs the *exact* value †; and it

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tended through the whole duration of the lives, or any great number of years; but when it is limited to a ſmall number of years, this term becomes conſiderable and ſhould by no means be omitted.

* If *a* and *c* are made to ſignify the complements of the lives of A and C for their *whole duration*, the anſwers may in ſome caſes be a *quarter* or even a *third* wrong.

† The terms of the ſeries in the Example to the Problem, ſuppoſing the number of years to be only *three*, and the decrements of life as they are in Table 3d, will be

$$\frac{1000\times428}{1.04\times435}\times\frac{8\times9}{2\times284\times365}+\frac{1000\times421}{\overline{1.04}|^{2}\times435}\times\overline{\frac{8\times8}{284\times365}+\frac{8\times8}{2\times284\times365}}+\frac{1000\times414}{\overline{1.04}|^{3}\times435}\times\overline{\frac{16\times8}{284\times365}+\frac{8\times8}{2\times284\times365}}=.3285+.8285+1,306=2.453\,l.$$

- - - - The ſame value by the approxiit will be eaſier to obtain an anſwer by finding the number it expreſſes, than by making uſe of the Rule.

Theſe obſervations are to be applied to all the ſolutions of Problems for determining the values of Reverſions depending on Survivorſhips within a given time.

Problem Thirty-first.—Page 188.—Let the ſame ſymbols be retained, and P denote the perpetuity. Let A, B, and C alſo ſignify the probabilities reſpectively that A, B and C ſhall live *t* years. - - - - - The values of the 1ſt, 2d, 3d, &c. years rents in this Problem may be conſidered as depending on one or other of the four following events. - - - Firſt, that A is dead, and B and C are both alive at the beginning of the 2d, 3d, 4th, &c. years. - - - Secondly, that A and B are both dead (A having died firſt) and C alive at the beginning of thoſe years. Thirdly, that A and C are both dead (A having died firſt) and B alive at the beginning of thoſe years. Laſtly, that A, B, and C are all dead at the beginning of thoſe years, A having died firſt.—By reaſoning in the ſame manner as hath been done in inveſtigating the 23d and the following Problems, the values of the ſeveral rents for *t* years and ever afterward on the *firſt* contingency

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mations in Problems 2d and 3d, and computing agreeable to the general Rule, may be found equal to 2.597*l*. Thoſe anſwers agree with each other very nearly in the preſent inſtance, owing to the decrements of life being *equal* in the Table. When this is not the caſe, the difference in ſo ſmall a term will be much more conſiderable.

will be expreſſed by the ſeries $\frac{e}{lr} \times \frac{o}{n} \times \frac{D}{L} + \frac{f}{lr^2} \times \frac{p}{n} \times \frac{D' + D''}{L} + \frac{g}{lr^3} \times \frac{q}{n} \times \frac{D' + D'' + D'''}{L}$ &c. (t) $+ \frac{B \times C \times \overline{1 - A}}{r^t} \times P$ ------ The values of the ſeveral rents for t years and ever afterward on the *ſecond* contingency will be expreſſed by the ſeries $\frac{e}{lr} \times \frac{D' \times \Delta'}{2Ln} + \frac{f}{lr^2} \times \frac{\overline{D' + D''} \times \overline{\Delta' + \Delta''}}{2Ln} + \frac{g}{lr^3} \times \frac{\overline{D' + D'' + D'''} \times \overline{\Delta' + \Delta'' + \Delta'''}}{2Ln}$ &c. ---- (t) $+ \frac{C \times \overline{1 - A} \times \overline{1 - B} \times P}{2r^t}$. ------- On the *third* contingency by $\frac{o}{nr} \times \frac{D' \times d'}{2Ll} + \frac{p}{nr^2} \times \frac{\overline{D' + D''} \times \overline{d' + d''}}{2Ll} + \frac{q}{nr^3} \times \frac{\overline{D' + D' + D'''} \times \overline{d' + d'' + d'''}}{2Ll}$ &c. (t) $+ \frac{B \times \overline{1 - C} \times \overline{1 - A}}{2r^t} \times P$.--On the *fourth* contingency by $\frac{D' \times d' \times \Delta'}{3nLlr} + \frac{\overline{D' + D''} \times \overline{d' + d''} \times \overline{\Delta' + \Delta''}}{3nLlr^2} + \frac{\overline{D' + D'' + D'''} \times \overline{d' + d'' + d'''} \times \overline{\Delta' + \Delta'' + \Delta'''}}{3nLlr^3}$ &c. (t) $+ \frac{\overline{1 - A} \times \overline{1 - B} \times \overline{1 - C}}{3r^t} \times P$.

The ſum of all theſe ſerieſes expreſſes the *exact* value according to any Table of obſervations, and gives the only juſt ſolution when the term does not exceed five or ſix years. When the term is longer, the following approximation will give anſwers ſufficiently accurate.

Let a, b, and c repreſent, as before, the complements of the lives of A, B, and C *for the given term* agreeable to any Table of obſervations. Then the value of the ſeveral rents for t years and ever afterward will be nearly according to that Table; ---- On the *firſt* contingency $\frac{\overline{b-1}.\,\overline{c-1}.\,1}{abcr} + \frac{\overline{b-2}.\,\overline{c-2}.\,2}{abcr^2} + \frac{\overline{b-3}.\,\overline{c-3}.\,3}{abcr^3}$ + &c. (t) $+ \frac{\overline{b-t}.\,\overline{c-t}.\,t}{abcr^t} \times$ P. ---- On the *ſecond* contingency $\frac{\overline{c-1}.\,1^2}{2abcr} + \frac{\overline{c-2}.\,2^2}{2abcr^2} + \frac{\overline{c-3}.\,3^2}{2abcr^3} +$ &c. (t) $+ \frac{\overline{c-t}.\,tt}{2abcr^t} \times$ P. ------ On the *third* contingency $\frac{\overline{b-1}.\,1^2}{2abcr} + \frac{\overline{b-2}.\,2^2}{2abcr^2} + \frac{\overline{b-3}.\,3^2}{2abcr^3}$ + &c. (t) $+ \frac{\overline{b-t}.\,tt}{2abcr^t} \times$ P. ----- On the *fourth* contingency $\frac{1^3}{3abcr} + \frac{2^3}{3abcr^2} + \frac{3^3}{3abcr^3}$ &c. ------ (t) $+ \frac{t^3}{3abcr^t} \times$ P. ------ If all theſe ſerieſes be reſolved and added together, they will be transformed into $\overline{\frac{1}{ar} + \frac{2}{ar^2} + \frac{3}{ar^3} \text{-----} (t)}$

$$\overline{+\frac{t}{ar^t}} \times P - \frac{\overline{b+c}}{2\,abc} \times \overline{\frac{1^2}{r} + \frac{2^2}{r^2} + \frac{3^2}{r^3} \text{--} (t)}$$

$$\overline{+\frac{tt}{r^t} \times P} + \frac{1}{3\,abc} \times \overline{\frac{1^3}{r} + \frac{2^3}{r^2} + \frac{3^3}{r^3} \&c.}$$

$\overline{(t) + \frac{t^3}{r^t}} \times P$.-----The firſt ſeries is the ſame with what Dr. *Price* has given in Note (G) of the Appendix to his Treatiſe on Reverſionary Payments, for determining the value of an Annuity after the deceaſe of A, ſhould it happen in a given term, and which has been alſo given in words in the ſolution of the 9th Problem in the preceding chapter of this work.------Let this ſeries, therefore, be denoted by R, the value of an Annuity on the longeſt of two equal lives (whoſe complement is *t*) by D, and the value of an Annuity on the longeſt of three ſuch equal lives by T; then the whole value of the Reverſion will be expreſſed by $R - \frac{\overline{b+c} \times tt}{2\,abc} \times \overline{P-D} + \frac{t^3}{3\,abc} \times \overline{P-T} = R - \overline{\frac{\overline{b+c}}{2} \times \overline{P-D} - \frac{t}{3} \times \overline{P-T}} \times \frac{tt}{abc}$, which is the general rule in page 188.

Having given theſe ſpecimens, it can hardly be neceſſary to proceed through the inveſtigations of the following Problems: I ſhall, therefore, give the serieſes expreſſing the *exact* value according to any Table of obſervations, and alſo the general theorem or *approximation*, without entering into any particular detail.

Problem Thirty-second, page 192.---The *exact* anſwer to this Problem is expreſſed by

$$\frac{e}{lr}\times\frac{D'\times\Delta'}{2\,Ln}+\frac{f}{lr^2}\times\frac{\overline{D'+D''}\times\overline{\Delta'+\Delta''}}{2\,Ln}+$$

$$\frac{g}{lr^3}\times\frac{\overline{D'+D''+D'''}\times\overline{\Delta'+\Delta''+\Delta'''}}{2\,Ln}\ \&c.\ (t)$$

$$+\frac{C\times\overline{1-A}\times\overline{1-B}}{2r^4}\times P\text{------}+\frac{o}{nr}$$

$$\times\frac{D'\times d'}{2Ll}+\frac{p}{nr^2}\times\frac{\overline{D'+D''}\times\overline{d'+d''}}{2\,Ll}+$$

$$\frac{q}{nr^3}\times\frac{\overline{D'+D''+D'''}\times\overline{d'+d''+d'''}}{2\,Ll}\ \&c.\ (t)$$

$$+\frac{B\times\overline{1-A}\times\overline{1-C}}{2\,r^4}\times P\text{---------}+$$

$$\frac{D'\times\Delta'\times d'}{3\,Llnr}+\frac{\overline{D'+D''}\times\overline{\Delta'+\Delta''}\times\overline{d'+d''}}{3\,Llnr^2}$$

$$+\ \&c.\ (t)+\frac{\overline{1-A}\times\overline{1-B}\times\overline{1-C}}{3\,Llnr^4}\times P.$$

------- The *approximation* as given in page 192 may be found equal to

$$\frac{tt}{abc}\times\overline{\frac{\overline{b+c}}{2}\times\overline{P-D}-\frac{2t.\overline{P-T}}{3}}.$$

Problem Thirty-third, page 193. ------ The *exact* anſwer by any Table is expreſſed by $\frac{D'\times\Delta'\times d'}{3nLlr} + \frac{\overline{D'+D''}\times\overline{\Delta'+\Delta''}\times\overline{d'+d''}}{3nLlr^2} + \frac{\overline{D'+D''+D'''}\times\overline{d'+d''+d'''}\times\overline{\Delta'+\Delta''+\Delta'''}}{3nLlr^3}$ &c. (t) + $\frac{\overline{1-A}\times\overline{1-B}\times\overline{1-C}}{3r^t}\times P$.

The general theorem or *approximation* is $= \frac{t^3}{3abc}\times\overline{P-T}$.

Problem Thirty-fourth, page 195. ----- The *exact* value is expreſſed by $\frac{e}{lr}\times\frac{D'\times\Delta'}{Ln} + \frac{f}{ln^2}\times\frac{\overline{D'+D''}\times\overline{\Delta'+\Delta''}}{Ln} + \frac{g}{lr^3}\times\frac{\overline{D'+D''+D'''}\times\overline{\Delta'+\Delta''+\Delta'''}}{Ln}$ &c. (t) + $\frac{C\times\overline{1-A}\times\overline{1-B}}{r^t}\times P + \frac{D'\times\Delta'\times d'}{3Lnlr} + \frac{\overline{D'+D''}\times\overline{\Delta'+\Delta''}\times\overline{d'+d''}}{3Lnlr^2}$ &c. (t) + $\frac{\overline{1-A}\times\overline{1-B}\times\overline{1-C}}{3r^t}\times P$ ------ The general theorem or *approximation* may be found equal to $\frac{tt}{ab}\times\overline{P-D} - \frac{2t^3}{3abc}\times\overline{P-T}$.

Problem Thirty-ninth, page 201.---- The *exact* anſwer to this Problem is expreſſed by the ſingle ſeries $\frac{D' \times \Delta' \times d'}{6 L n l r} + \frac{\overline{D' + D''} \times \overline{\Delta' + \Delta''} \times \overline{d' + d''}}{6 L n l r^2}$ &c. (t) $+ \frac{\overline{1 - A} \times \overline{1 - B} \times \overline{1 - C}}{6 r^t} \times P$ ... and the *approximation* to the real value by $\frac{t^3}{6 abc} \times \overline{P - T}$ which is juſt *half* the value of the Reverſion by Problem 33d.

Problem Fortieth, page 202.--- The ſeries expreſſing the anſwer to this Problem is $\frac{1}{2 L l} \times \overline{\frac{o \times D' \times d'}{nr} + \frac{p \times \overline{D' + D''} \times \overline{d' + d''}}{nr^2} \text{ \&c. } (t)}$ $+ \overline{1 - A} \times \overline{1 - C} \times \frac{V}{2}$; V denoting the value of the life of B after t years.

The general theorem or *approximation* may be found $= \frac{tt}{2ac} \times \overline{M - D}$; M repreſenting the value of an Annuity on the longeſt of three lives, one of whom being of the age of B, and the two others of a life whoſe complement is t, and whoſe decrements are equal.

Corollary to Problem Fortieth.--- The *exact* anſwer is evidently the firſt part of the ſeries given above, or $\frac{1}{2 L l} \times \overline{\frac{o \times D' \times d'}{nr}} +$

$\frac{p \times \overline{D' + D''} \times \overline{d' + d''}}{nr^2}$ ----- (*t*), and the *approximation* to this value may be found equal to $\frac{tt}{ac} \times \overline{\frac{\overline{\tau - D}}{2} - \frac{t \times \overline{\tau - T}}{2b}}$; τ denoting the value of an Annuity certain for *t* years.

Problem Forty-first, page 205. ---- The value of this Annuity being evidently the value in Problem 40th, ſubtracted from the Reverſion on the life of B *after* A, provided B ſhould ſurvive A in the given time; the only thing neceſſary for the ſolution of this Problem is to find the value of ſuch a Reverſion. ----- The ſeries expreſſing the value of this Reverſion is $\frac{o}{nr} \times \frac{D'}{L} + \frac{p}{nr^2} \times \frac{D' + D''}{L} + \frac{q}{nr^3} \times \frac{D' + D'' + D'''}{L}$ &c. (*t*) $+ \overline{1 - A} \times V$, which, therefore, being leſſened by the value found from the *ſeries* in Problem 40th, will give the anſwer in *exact* agreement to any Table of obſervations; V denoting the value of the life of B after *t* years as in that Problem.

The *approximation* to the real value of the Reverſion juſt mentioned (and which is to be leſſened by the value found from the *theorem* in Problem 40th) is equal to $\frac{t}{a} \times \overline{B - \beta}$, where B repreſents the value of the life of B, and β the value of an Annuity on the joint continuance of B and another life whoſe complement is t, and whoſe decrements are equal.

PROBLEM FORTY-SECOND, page 208. - - - The ſerieſes expreſſing the *exact* value in this Problem are

$$\frac{e}{lr} \times \frac{D' \times \Delta'}{Ln} + \frac{f}{ln^2} \times \frac{\overline{D' + D''} \times \overline{\Delta' + \Delta''}}{Ln}$$

$$+ \frac{g}{lr^3} \times \frac{\overline{D' + D'' + D'''} \times \overline{\Delta' + \Delta'' + \Delta'''}}{Ln} \text{ \&c. } (t)$$

$$+ \frac{C \times \overline{1 - A} \times \overline{1 - B}}{r^t} \times P \text{ - - - - - - and}$$

$$\frac{D' \times \Delta' \times d'}{2Llnr} + \frac{\overline{D' + D''} \times \overline{\Delta' + \Delta''} \times \overline{d' + d''}}{2Llnr^2}$$

$$\text{\&c. } (t) + \frac{\overline{1 - A} \times \overline{1 - B} \times \overline{1 - C}}{2r^t} \times P.$$

The general theorem, or *approximation* to the exact value, is $\frac{tt}{ab} \times \overline{P - D} - \frac{t^3}{2abc} \times \overline{P - T}$.

Note

Note (K).

An Account of the Method by which the Values in Annual *Payments have been deduced.*

PROBLEM XVIII. page 144. - - - - This claim will be undetermined as long as the three joint lives A, B, and C continue, and alſo during all that may happen to remain of the joint lives of A and B after C. - - - Now the laſt of theſe values is found by ſubtracting the value of the three joint lives A, B, and C from the joint lives in expectation; that is, from the joint lives of A and B - - - Add to this the former value, and the reſult will give the general rule. - - - - Let A B C, A C, B C, A B, conſtantly repreſent the values of the ſeveral *joint* lives combined as thoſe letters are; and let $\dot{A}\,\dot{C}$, $\dot{B}\,\dot{C}$, $\dot{A}\,\dot{B}\,\dot{C}$, &c. repreſent the values of the *longeſt* of two or three lives. Then in the preſent caſe we ſhall have $AB - ABC + ABC = AB$ for the diviſor. - - - - Should the money *become payable*, it will become ſo at the ſame time that the *claim is determined*; and, therefore, the annual payments in both caſes are the ſame.

The like may be obſerved in all the following ſolutions, where no mention is made of any difference in thoſe values.

PROBLEM XX. page 149. - - - - - This claim cannot be determined till after the extinction of the joint lives of A and C, and likewiſe the joint lives of A and B. In other words, it will be undetermined while the joint lives of A and C continue, and alſo during the whole time that the joint lives of A and B may happen to continue after C. The laſt of theſe values is equal to $\overrightarrow{AB - ABC}$, which added to the firſt value, or AC, we have $AC + AB - ABC$ for the required diviſor.

Problem XXI. page 151.----The time when the claim *is determined* is found juſt in the ſame manner as in the preceding Problem, and conſequently the diviſor for aſcertaining the annual payments in this caſe will be alſo $= AC + AB - ABC$.

Should the claim, however, be determined in favour of the purchaſer, it is evident that the money will not become payable till after a further time, and, therefore, under theſe circumſtances the annual payments ought to be *leſs*; for they muſt be continued not only during the joint lives of A and C and the remainder of the joint lives of A and B after C, but likewiſe during all that may happen to remain of the ſingle life A after the longeſt of the two lives C and B. Now the value of an Annuity on the laſt of theſe Reverſions is $= \overset{\prime}{A}\overset{\prime}{B}\overset{\prime}{C} - \overset{\prime}{B}\overset{\prime}{C}$, which being reſolved * and then added to $AC + AB - ABC$, given above, will at laſt be found equal to an Annuity on the ſingle life of A.

---

* By Scholium to Problem 6th, $\overset{\prime}{A}\overset{\prime}{B}\overset{\prime}{C} = A + B + C + ABC - AB - AC - BC$ ----- By the Rule in page 55, $\overset{\prime}{B}\overset{\prime}{C} = B + C - BC$; hence $\overset{\prime}{A}\overset{\prime}{B}\overset{\prime}{C} - \overset{\prime}{B}\overset{\prime}{C} = A + ABC - AC - AB$, which being added to $AC + AB - ABC$, we have A for the required diviſor.

Problem XXII. page 153. ------ Let A be always ſuppoſed to be the youngeſt of the two lives A and B. ----- In this caſe the claim will be undetermined during the joint lives of B and C, and alſo during the continuance of the joint lives of A and C after B. Hence the required diviſor is equal to BC + AC — ABC.

Problem XXIII. page 154. ---- It is evident that the claim is *determined* here at the extinction of the three joint lives. But the annual payments till the claim *becomes due*; that is, till the claim is determined and afterwards till it becomes payable; provided it ſhould be determined in favour of the purchaſer, will be much *leſs* than the preceding ones, ſince they muſt be continued not only during the joint lives of A, B and C, but alſo during the remainder of the longeſt of the two lives A and B after C. ---- The value of an Annuity on ſuch a Reverſion is equal $\overset{\prime}{A}\overset{\prime}{B}\overset{\prime}{C} - C$; therefore the diviſor in this caſe will be $= ABC + \overset{\prime}{A}\overset{\prime}{B}\overset{\prime}{C} - C$.

Problem XXIV. page 161. ---- The claim in this Problem cannot be determined till after the extinction of the joint lives of A and C, and all that may happen to remain of the joint lives of A and B after C. Conſequently the required diviſor will be = AC + AB — ABC.

PROBLEM XXV. page 163. ---- The time in which the claim, in this caſe, is *determined*, may be found by reaſoning in the ſame manner as in Problems 20th and 21ſt. Therefore the general diviſor will alſo be the ſame; that is, equal to AC + AB — ABC.

The time in which the claim *becomes payable* may likewiſe be found as in Problem 21ſt. But in that, as well as in the preſent Problem, this time may be aſcertained in a much ſhorter way; for ſince it is evident that the claim cannot be determined *againſt* a purchaſer by any other death than that of A, it follows, by making the annual payments during A's life, that a purchaſer cannot become bound to make any payments *after* the claim has been determined againſt him.

PROBLEM XXVI. page 164. ---- The *determination* of the claim in this Problem depends on the ſame contingencies with thoſe in Problem 20th; for it is plain that whenever it is determined that A *will* or *will not* be the *ſecond* that dies, it will at the ſame time be determined that he *will not* or *will* be either the *firſt* or *laſt* that dies.

The ſame reaſoning may be applied, towards finding the value in *annual* payments till the money *becomes due*, as in Problem 25th.

It may not be improper in this place to add the following inveſtigation; which will further confirm the Rule given in Problems 20th, 21ſt, 25th, and 26th, for finding the value in annual payments till the claim is *determined.* - - - - - Let the preſent, or 26th, Problem be taken. - - - - - The title to the 1ſt, 2d, 3d, &c. payments of an Annuity to be continued till the claim, ſpecified in this Problem, is *determined*, depends on three different contingencies. — 1ſt, On A, B, and C being all living at the end of the 1ſt, 2d, 3d, &c. years. Secondly, On A and B being living and C dead at the end of the 1ſt, 2d, 3d, &c. years. Thirdly, On A and C being living and B dead at the end of the 1ſt, 2d, 3d, &c. years. The value, therefore, of the Annuity, as far as it depends on the *firſt* of theſe events, is the value of the three joint lives. The value of the Annuity, as far as it depends on the *ſecond* event, is the value of the two joint lives A and B after C; that is, the value of the two joint lives A and B leſſened by the value of the three joint lives. The third value is the value of the two joint lives A and C after B, or the value of the joint lives A and C leſſened by the three joint lives. The ſum of all theſe values, or AC+AB—ABC, is the rule.

PROBLEM XXVIII. page 173. - - - - The claim in this Problem will be *determined* at the extinction of the joint lives of A and C; but the money will not *become payable* till after the *longeſt* of the two lives A and B: nor can the purchaſer even then be entitled to any claim, unleſs A ſhall have died in the life-time of C. In order, therefore, to obtain a proper diviſor in this caſe, it will be neceſſary to add to the value of the joint lives

A and C the value of the life of B after A, provided C ſhall have ſurvived A. - - - - The laſt of theſe values is given in Problem 27th, which has been inveſtigated partly with a view to the ſolution of this Problem.

Problem XXIX. page 175. - - - - The method of finding the time in which the claim is here *determined*, is nearly the ſame with that which has been uſed in Problem 18th. For an Annuity to be paid till this event is known muſt be paid during the three joint lives, and alſo during the joint lives of B and C after A ; that is, the value of ſuch an Annuity is equal to ABC + BC — ABC = BC. But as the purchaſer cannot come into *poſſeſſion* of the eſtate (if the claim is determined in his favour) till after the extinction of the laſt ſurviving life C ; it follows that the value of an Annuity to be continued to this period muſt be equal to the former value increaſed by the value of the life of C after B, provided B ſurvives A, which is found by the Corollary to Problem 27th *.

* The rules, for finding the values of two lives after one, of one life after two, and of two *joint* lives after one, which have been uſed in the preceding ſolutions, are the ſame with thoſe in the 16th, 18th, and 19th Problems in Mr. *Simpſon*'s Select Exerciſes.

TABLE FIRST.

The preſent Value of 1 *l. to be received at the End of any Number of Years, not exceeding* 92; *diſcounting at the Rate of* Four per Cent. *Compound Intereſt.*

Years.	Four per Cent.	Years.	Four per Cent.
1	.961538	28	.333477
2	.924556	29	.320651
3	.888996	30	.308319
4	.854804	—	—
5	.821927	31	.296460
6	.790315	32	.285058
7	.759918	33	.274094
8	.730690	34	.263552
9	.702587	35	.253415
10	.675564	36	.243669
—	—	37	.234297
11	.649581	38	.225285
12	.624597	39	.216621
13	.600574	40	.208289
14	.577475	—	—
15	.555265	41	.200278
16	.533908	42	.192575
17	.513373	43	.185168
18	.493628	44	.178046
19	.474642	45	.171198
20	.456387	46	.164614
—	—	47	.158283
21	.438834	48	.152195
22	.421955	49	.146341
23	.405726	50	.140713
24	.390121	—	—
25	.375117	51	.135301
26	.360689	52	.130097
27	.346817	53	.125093

Years.	Four per Cent.	Years.	Four per Cent.
54	.120282	74	.054895
55	.115656	75	.052784
56	.111207	76	.050754
57	.106930	77	.048801
58	.102817	78	.046924
59	.098963	79	.045120
60	.095060	80	.043384
61	.091404	81	.041716
62	.087889	82	.040111
63	.084508	83	.038569
64	.081258	84	.037085
65	.078133	85	.035659
66	.075128	86	.034287
67	.072238	87	.032968
68	.069460	88	.031700
69	.066788	89	.030481
70	.064219	90	.029309
71	.061749	91	.028182
72	.059374	92	.027098
73	.057091		

TABLE SECOND.

The preſent Value of an Annuity of One Pound, for any Number of Years not exceeding 92; *at the Rate of* Four per Cent.

Years.	Years Purchaſe.	Years.	Years Purchaſe.
1	.9615	28	16.6630
2	1.8860	29	16.9837
3	2.7750	30	17.2920
4	3.6298	—	———
5	4.4518	31	17.5884
6	5.2421	32	17.8735
7	6.0020	33	18.1476
8	6.7327	34	18.4111
9	7.4353	35	18.6646
10	8.1108	36	18.9082
—	———	37	19.1425
11	8.7604	38	19.3678
12	9.3850	39	19.5844
13	9.9856	40	19.7927
14	10.5631	—	———
15	11.1183	41	19.9930
16	11.6522	42	20.1856
17	12.1656	43	20.3707
18	12.6592	44	20.5488
19	13.1339	45	20.7200
20	13.5903	46	20.8846
—	———	47	21.0429
21	14.0291	48	21.1951
22	14.4511	49	21.3414
23	14.8568	50	21.4821
24	15.2469	—	———
25	15.6220	51	21.6174
26	15.9827	52	21.7475
27	16.3295	53	21.8726

Years.	Years Purchaſe.	Years.	Years Purchaſe.
54	21.9929	74	23.6276
55	22.1086	75	23.6804
56	22.2198	76	23.7311
57	22.3267	77	23.7799
58	22.4295	78	23.8268
59	22.5284	79	23.8720
60	22.6234	80	23.9153
61	22.7148	81	23.9571
62	22.8027	82	23.9972
63	22.8872	83	24.0357
64	22.9685	84	24.0728
65	23.0466	85	24.1085
66	23.1218	86	24.1428
67	23.1940	87	24.1757
68	23.2635	88	24.2074
69	23.3302	89	24.2379
70	23.3945	90	24.2672
71	23.4562	91	24.2954
72	23.5156	92	24.3225
73	23.5727		

TABLE THIRD.

Shewing the Probabilities of the Duration of Life, as deduced by Dr. PRICE *from Obſervations on the Bills of Mortality at* NORTHAMPTON.

Ages.	Perſons living.	Decrem. of Life.	Ages.	Perſons living.	Decrem. of Life.
1	849	127	30	435	7
2	722	50	31	428	7
3	672	26	32	421	7
4	646	21	33	414	7
5	625	16	34	407	7
6	609	13	35	400	7
7	596	10	36	393	7
8	586	9	37	386	7
9	577	7	38	379	7
10	570	6	39	372	7
11	564	6	40	365	8
12	558	5	41	357	8
13	553	5	42	349	8
14	548	5	43	341	8
15	543	5	44	333	8
16	538	5	45	325	8
17	533	5	46	317	8
18	528	6	47	309	8
19	522	7	48	301	8
20	515	8	49	293	9
21	507	8	50	284	9
22	499	8	51	275	8
23	491	8	52	267	8
24	483	8	53	259	8
25	475	8	54	251	8
26	467	8	55	243	8
27	459	8	56	235	8
28	451	8	57	227	8
29	443	8	58	219	8

Ages.	Perſons living.	Decrem. of Life.	Ages.	Perſons living.	Decrem. of Life.
59	211	8	76	75	8
60	203	8	77	67	7
61	195	8	78	60	7
62	187	8	79	53	7
63	179	8	80	46	7
64	171	8	81	39	7
65	163	8	82	32	6
66	155	8	83	26	5
67	147	8	84	21	4
68	139	8	85	17	4
69	131	8	86	13	3
70	123	8	87	10	2
71	115	8	88	8	2
72	107	8	89	6	2
73	99	8	90	4	2
74	91	8	91	2	1
75	83	8	92	1	1

TABLE FOURTH.

Shewing the Expectations *of Life in* NORTHAMPTON, *according to the preceding Table.*

Age.	Expectation.	Age.	Expectation.	Age.	Expectation.	Age.	Expectation.
1	33.29	24	31.26	47	19.38	70	8.42
2	38.06	25	30.78	48	18.87	71	7.97
3	39.86	26	30.30	49	18.38	72	7.53
4	40.44	27	29.82	50	17.94	73	7.10
5	40.78	28	29.34	51	17.51	74	6.68
6	40.84	29	28.86	52	17.02	75	6.28
7	40.72	30	28.38	53	16.53	76	5.90
8	40.41	31	27.83	54	16.05	77	5.54
9	40.03	32	27.29	55	15.56	78	5.13
10	39.52	33	26.74	56	15.07	79	4.74
11	38.93	34	26.19	57	14.58	80	4.39
12	38.34	35	25.65	58	14.10	81	4.09
13	37.69	36	25.09	59	13.61	82	3.87
14	37.02	37	24.54	60	13.13	83	3.65
15	36.36	38	23.98	61	12.65	84	3.40
16	35.69	39	23.42	62	12.17	85	3.08
17	35.02	40	22.86	63	11.69	86	2.88
18	34.35	41	22.37	64	11.24	87	2.60
19	33.74	42	21.87	65	10.74	88	2.12
20	33.19	43	21.37	66	10.27	89	1.66
21	32.71	44	20.87	67	9.80	90	1.25
22	32.22	45	20.37	68	9.34	91	1.00
23	31.74	46	19.87	69	8.88	92	.50

TABLE FIFTH.

Shewing the preſent Value of an Annuity of 1 *l. on a ſingle Life at* Four *per Cent., according to Mr.* De Moivre's *Hypotheſis; and nearly according to the Probabilities of Life at* NORTHAMPTON. *See Table, Page* 64.

Age.	Years Purchaſe.	Age.	Years Purchaſe.	Age.	Years Purchaſe.	Age.	Years Purchaſe.
8	16.791	28	14.946	48	11.748	68	6.714
9	16.882	29	14.816	49	11.548	69	6.394
10	16.882	30	14.684	50	11.344	70	6.065
11	16.791	31	14.549	51	11.135	71	5.728
12	16.698	32	14.411	52	10.921	72	5.383
13	16.604	33	14.270	53	10.702	73	5.029
14	16.508	34	14.126	54	10.478	74	4.666
15	16.410	35	13.979	55	10.248	75	4.293
16	16.311	36	13.829	56	10.014	76	3.912
17	16.209	37	13.676	57	9.773	77	3.520
18	16.105	38	13.519	58	9.527	78	3.111
19	15.999	39	13.359	59	9.275	79	2.707
20	15.891	40	13.196	60	9.017	80	2.284
21	15.781	41	13.028	61	8.753	81	1.850
22	15.669	42	12.858	62	8.482	82	1.406
23	15 554	43	12.683	63	8.205	83	0.950
24	15.437	44	12.504	64	7.921	84	0.481
25	15.318	45	12.322	65	7.631	85	0.000
26	15.197	46	12.135	66	7.333		
27	15.073	47	11.944	67	7.027		

TABLE SIXTH*.

Shewing the preſent Value of an Annuity of 1 *l. on the joint Continuance of two Lives at* Four *per Cent., according to Mr.* De Moivre's *Hypotheſis; and nearly according to the Probabilities of Life at* NORTHAMPTON. *See Table, Page* 74.

Ages.		Years Purchaſe.	Ages.		Years Purchaſe.
10	10	13.342	15	70	5.623
	15	13.093	-		
	20	12.808	20	20	12.341
	25	12.480		25	12.051
	30	12.102		30	11.711
	35	11.665		35	11.314
	40	11.156		40	10.847
	45	10.564		45	10.297
	50	9.871		50	9.648
	55	9.059		55	8.879
	60	8.105		60	7.967
	65	6.980		65	6.882
	70	5.652		70	5.590
			-		
15	5	12.860	25	25	11.786
	20	12.593		30	11.468
	25	12.281		35	11.095
	30	11.921		40	10.655
	35	11.501		45	10.131
	40	11.013		50	9.509
	45	10.440		55	8.766
	50	9.767		60	7.880
	55	8.975		65	6.826
	60	8.041		70	5.551
	65	6.934	-		

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* This is taken from Dr. *Price*'s Table of the values of joint lives publiſhed in his *Treatiſe on Reverſionary Payments*. See page 328, 3d edit.

Ages.		Years Purchase.
30	30	11.182
	35	10.838
	40	10.428
	45	9.936
	50	9.345
	55	8.634
	60	7.779
	65	6.748
	70	5.505
35	35	10.530
	40	10.157
	45	9.702
	50	9.149
	55	8.476
	60	7.658
	65	6.662
	70	5.450
40	40	9.826
	45	9.418
	50	8.911
	55	8.283
	60	7.510
	65	6.556
	70	5.383
45	45	9.063
	50	8.619
	55	8.044
	60	7.332
	65	6.425
	70	5.300
50	50	8.235
	55	7.738
	60	7.091
	65	6.258
	70	5.193
55	55	7.332
	60	6.781
	65	6.036
	70	5.053
60	60	6.351
	65	5.730
	70	4.858
65	65	5.277
	70	4.571
70	70	4.104

TABLE

TABLE SEVENTH*.

Shewing the preſent Value of an Annuity of 1 *l. on the joint Continuance of* Three *equal Lives at* Four per Cent., *according to the Probabilities of Life in* NORTHAMPTON. *See Table 3d.*

Common Age.	Years Purchaſe.	Common Age.	Years Purchaſe.
60	4.7826	76	1.9089
61	4.6115	77	1.7846
62	4.4382	78	1.5843
63	4.2626	79	1.3906
64	4.0849	80	1.2121
65	3.9050	81	1.0685
66	3.7230	82	1.0117
67	3.5390	83	0.9617
68	3.3533	84	0.8981
69	3.1662	85	0.7606
70	2.9780	86	0.7690
71	2.7895	87	0.7568
72	2.6015	88	0.5368
73	2.4160	89	0.3233
74	2.2352	90	0.1346
75	2.0636	91	0.1202

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* This table has been computed agreeable to the method deſcribed in chap. II. ſect. II.; and as it was only deſigned for the purpoſe of comparing the values in ſome of the foregoing examples, (where the hypotheſis and the approximations in Prob. 5th and 6th are uſed) with the *exact* values, it was thought unneceſſary to extend it below the age of 60; eſpecially as the hypotheſis and approximations correſpond ſo nearly with the truth under that age. See Schol. to Prob. 6th, page 102.

ESSAY,

CONTAINING AN

ACCOUNT

OF THE

ROGRESS from the REVOLUTION,

AND THE

PRESENT STATE,

OF

OPULATION

IN

ENGLAND AND *WALES*.

y RICHARD PRICE, D. D. and F. R. S.

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ASTOR, [illegible]
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OBSERVATIONS ON The Progreſs (ſince the Revolution) and the preſent State of Population in ENGLAND and WALES.

IT will be proper to introduce theſe obſervations with the following accounts of LONDON and MIDDLESEX.

Number of Houſes in LONDON, SOUTHWARK, WESTMINSTER, *and the* COUNTY OF MIDDLESEX, *in the Year* 1777; *from the Accounts of the Surveyors of the Houſe and Window Duties.*

Houſes charged in 1777, having 25 windows and upwards — —	12,560
Houſes charged, having leſs than 25 windows — — —	61,080
Total of houſes charged — —	73,640
Uninhabited houſes chargeable —	3,368
Total of houſes charged and chargeable — — — —	77,008
Cottages not charged by reaſon of poverty — — —	13,562
Total of houſes — —	90,570

Number of Houſes in London, Southwark, Weſtminſter, *and the* County of Middleſex, *from the Survey mentioned by Dr.* Brackenridge *in a Paper read to the Royal Society in March* 1758, *and publiſhed in the Philoſophical Tranſactions, vol.* 50, *p.* 471.

Houſes charged to the houſe and window tax in 1757 — —	63,480
Houſes uninhabited — —	4,810
Total of houſes charged and chargeable — —	68,290
Cottages — — —	19,324
Total of houſes, including cottages — — —	87,614

REMARKS.

Theſe accounts ſhew, that the number of houſes in *London*, *Weſtminſter*, *Southwark*, and *all Middleſex* had, in the courſe of about 20 years preceding 1778, increaſed 2,956 in the whole; but that the houſes excuſed on account of poverty had decreaſed 5,762; from whence it follows, that the houſes *charged* and *chargeable* had increaſed 8,718. —It ſhould be conſidered, that moſt probably this is leſs than the real increaſe of the beſt ſort of houſes; for the decreaſe of the cottages proves, that the meaneſt of the houſes[*] which pay the tax muſt likewiſe have decreaſed; and this decreaſe is to be added to 8,718, in order to obtain the whole increaſe of the beſt houſes; for it is obvious, that, if the beſt houſes had not increaſed as much as the worſt decreaſed, the total of houſes, inſtead of being greater in 1777, muſt have been leſs.—Perhaps, therefore, we ſhall reckon moderately enough if we reckon an increaſe within the laſt 20 years of 10,000 ſubſtantial houſes in and about *London*; and this is a number that falls little ſhort of the whole number of houſes in *Liverpool* and *Mancheſter*.

The increaſe of buildings in *London* has for ſeveral years been the object of general obſervation. It deſerves particular notice that it is derived entirely from the increaſe of luxury; an evil which, while it flatters, never fails to deſtroy. It has been ſhewn from authentic accounts, that the decreaſe of the lower people in *London* and *Middleſex* has kept pace with the increaſe of buildings. The annual deaths alſo in the Bills of Mortality have for many years been decreaſing, and are now near 6,000 *per annum* leſs than they were fifty years ago. In particular; it is obſervable with reſpect to that part of *London* which lies within the city walls, that, though always filled with houſes, the births and burials, and, conſequently, the inhabitants *, have decreaſed ONE HALF.—The juſt account of this muſt be, that thoſe who cannot now ſatisfy themſelves without whole houſes, or, perhaps, two or three houſes, to live in, uſed formerly to be ſatisfied with lodgings, or with parts of houſes.

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[*] That is, houſes paying the houſe duty of 3s. only. The number of theſe houſes in 1777 was 5,738; but I have no account of it for any preceding year. It will appear preſently, that taking *England* in the groſs, there has been a great decreaſe in theſe houſes; and this makes it almoſt certain they muſt have decreaſed in *Middleſex.* —The decreaſe of cottages, or houſes excuſed, ſince 1757, is the more remarkable, becauſe the houſe and window duties have been increaſed ſince that year by three different acts of parliament, the firſt in 1758, the ſecond in 1762, and the third in 1766.

The number of *houſes* in *London*, *Weſtminſter*, and all *Middleſex*, in 1690, was

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* See a particular account of this fact in my Obſervations on Reverſionary Payments, page 190, 3d edit.

111,215; according to Dr. *Davenant*'s account from the Hearth-books *.

I will only further obſerve concerning the preceding accounts, that they demonſtrate that the number of inhabitants in London has been greatly over-rated. They have been ſometimes eſtimated at a million. In an Eſſay on the State of London, on Population, &c. in the Treatiſe on Reverſionary Payments, I offered evidence, which I thought little ſhort of demonſtration, to prove that they fell ſhort of 651,000. But it now appears that, allowing 6 to a houſe †, and including the whole county of Middleſex, their number in 1777 was only 543,420.

That ſix to a houſe for London, and five to a houſe for all England, is too large an allowance, will be proved by the following recital of facts.

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* See Dr. Davenant's works, vol. 1ſt, page 38. This number does not include *Southwark*.——The average of burials for five years in *London* before the preſent year, or 1779, was 20,835. The average for five years before 1690 was 22,742; that is, conſiderably greater than it has been for the laſt five years, though twelve pariſhes, now the moſt populous, were not then included in the Bills.

In NORWICH, according to a ſurvey in 1752	Houſes,	7,139	Inhabitants,	36,169	To a houſe,	5.
Shrewſbury, by a ſurvey in 1750,		3,078		13,328		$4\frac{1}{3}$.
Northampton, by a ſurvey in 1746		1,083		5,136		$4\frac{3}{4}$.
The pariſh of Ackworth, Yorkſhire, in 1767		184		728		4.
Newbury, Berkſhire, n 1768		930		3,732		4.
Speen, adjoining to Newbury, in 1768		303		1,200		4.
Aldwinckle, Northamptonſhire, in 1772		96		402		$4\frac{1}{5}$.
The pariſh of Holy Croſs, near Shrewſbury, in 1760		242		1,050		$4\frac{1}{3}$.
Altringham, Cheſhire, in 1772		248		1,029		$4\frac{1}{7}$.
The Pariſh of St. Michael's, Cheſter, in 1772		127		618		$4\frac{5}{6}$.
The town and pariſh of Bala, North-Wales, in 1774		401		1,723		$4\frac{3}{10}$.
Fifty-nine Dutch villages mentioned by Struyk		12,005		45,888		$3\frac{5}{6}$.
Birmingham, in 1770		6,025		30,804		$5\frac{1}{9}$.
Liverpool, in 1773, including 400 in the Poor-houſe		6,340		34,407		$5\frac{2}{5}$.
Mancheſter and Salford, in 1773		4,338		27,246		$6\frac{1}{4}$.
Leeds, in 1775	Families	4,096		17,121	To a family	$4\frac{1}{6}$.

In the Diſtrict of Vaud in Switzerland	Families	25,778	Inhabitants,	112,951	To a family,	$4\frac{1}{3}$.
Cheſter, in 1774	Families	3,428	—	14,713	—	$4\frac{3}{10}$.
Rome, in 1770	Families	37,449	—	158,442	—	$4\frac{1}{4}$.
Calne, Wiltſhire	Families	776	—	3,467	—	$4\frac{1}{2}$.
Liverpool	Families	8,002	—	34,407	—	$4\frac{3}{10}$.
Mancheſter	Families	6,416	—	27,246	—	$4\frac{2}{5}$.
Bolton in Lancaſhire, in 1773, including Little Bolton	Houſes	1,178	—	5,339	To a houſe,	$4\frac{1}{2}$.
Bury in Lancaſhire, in 1772	Houſes	463	—	2,090	—	$4\frac{1}{2}$.
The pariſh of Bala in North-Wales, in 1774	Houſes	401	—	1,723	—	$4\frac{3}{10}$.
Chippenham, Wilts, in 1773	Houſes	483	—	2,407	—	5.
Brenhill, near Calne, in Wiltſhire	Houſes	218	—	1,206	—	$5\frac{1}{2}$.
The Iſland of Sicily (ſee end of 2d vol. of Brydone's Travels)	Houſes	268,120	—	1,123,163	—	$4\frac{1}{5}$.
Fourteen market towns mentioned by Dr. Short, Comparative Hiſtory, page 58	Families	20,371	—	97,611	To a family	$4\frac{3}{4}$.
Sixty-five country pariſhes, ibid.	Families	17,208	—	76,284	—	$4\frac{2}{5}$.

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In the Pariſh of *Skelton*, Yorkſhire, in 1777	Houſes	109	Inhabitants,	506	To a houſe,	$4\frac{1}{5}$.
The town and pariſh of Wycombe, Bucks	Families	500	—	2,461	To a family,	$4\frac{9}{10}$.
Worſley, Barton, Pendleton, Pendlebury, and Clifton, Lancaſhire, in 1778	Families	1,685	—	9,117	—	$5\frac{2}{5}$.
Pariſh of St. Cuthbert, Edinburgh, in 1743 (ſee Maitland's Hiſtory of Edinburgh, page 171)	Families	2,370	—	9,731	—	$4\frac{1}{10}$.
In a number of ſmall towns and pariſhes in the Generalities of Auvergne, Lyon, and Rouen, in France (ſee *Recherches ſur la Population*, *par* M. Meſſance, pages 8, 26, and 62)	Families	24,910	—	99,002	—	4.
Pariſh of Mancheſter, excluſive of the town, in 1774	Families	2,525	—	13,786	—	$5\frac{1}{2}$.
	Houſes	2,412	—	13,786	To a houſe,	$5\frac{7}{10}$.
Pariſh in the city of London (ſee Phil. Tract. part 2d, page 796)	—	—	—	—	6 to a houſe.	

Number of Houſes in England and Wales, from the Returns of the Surveyors of the Houſe and Window Duties in 1766 *and* 1777.

	In 1765.	In 1777.
Houſes charged, having 25 windows and upwards — — —	*32,595	32,595
Houſes having 21, 22, 23 and 24 windows —	12,404	14,623
Total of houſes having more than 20 windows	44,999	47,218
Houſes having from 12 to 20 windows —	88,494	98,756
Total of houſes having more than 11 windows	133,493	145,974
Houſes having 8, 9, 10, and 11 windows —	102,525	117,857
Total of houſes having more than 7 windows	236,018	263,831
Increaſe in 1777 of houſes having from 8 to 24 windows — — —		27,813

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[*] In the returns for 1765 this number is wanting. I have, therefore, ſuppoſed it the ſame that it was found to be in 1777. But the truth is, that it muſt have been leſs, as will appear preſently.

	In 1765.	In 1777.
Houſes charged having 7 windows	*	131,950
Total of houſes paying the window tax	236,018	395,781
Houſes paying only the houſe tax of 3*s.*	442,897	286,296
Total of houſes charged	678,915	682,077
Increaſe in 1777 of houſes charged		3,162
Houſes uninhabited, but chargeable	†25,628	19,396
Total of houſes charged and chargeable	704,543	701,473
Decreaſe in 1777 of houſes charged and chargeable		3,070
Cottages excuſed on account of poverty only	276,149	251,261
Total of houſes charged, chargeable, and excuſed	980,692	952,734

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* The number of houſes in 1765, having exactly ſeven windows, was 400,273; but by the law, as it then ſtood, all ſuch houſes were exempted from the window tax. In 1766 the tax was extended to theſe houſes; and the conſequence was, that near two thirds of them were reduced to houſes having only ſix windows.

† The decreaſe which may be here obſerved in the number of empty, but chargeable, houſes, is an effect which could not but attend the greater demand for houſes which produced the increaſe between 1765 and 1777, of houſes having more than ſeven windows.

Decreaſe of houſes charged, chargeable, and excuſed, from 1765 to 1777 — — — — —	27,958
To this decreaſe add the increaſe of houſes having from 8 to 24 windows, or — — — — —	27,813

And the total will ſhew, that the number of houſes not having *eight* windows was 55,771 leſs, in 1777, than it had been in 1765.

Again; from 27,813, the increaſe in 1777 of houſes having from 8 to 24 windows, ſubtract 3,162, the increaſe of houſes charged, having leſs than 25 windows; and it will appear, that in the houſes charged, having 7 windows or leſs, there has been in the ſame period a decreaſe of 24,651 houſes. ——But this is by no means the whole decreaſe of houſes of this ſort. The increaſe of houſes having more than 24 windows ought to be added; but the number of ſuch houſes not having been given in the return for 1766, it does not appear what this increaſe has been. It ſeems, however, paſt doubt, that there muſt have been ſuch an increaſe, becauſe all other houſes having more than ſeven windows had increaſed.

NUMBER OF HOUSES IN ENGLAND AND WALES *in* 1759, *from the Return of the Surveyors of the Houſe and Window Duties.*

Houſes charged in 1759	679,149
Uninhabited houſes in 1759 chargeable	24,904
Houſes excuſed on account of poverty only	282,429
Total of houſes in 1759	986,482
——— in 1765, *ſee p.* 286.	980,692
——— in 1777, *ſee p.* 286.	952,734
Diminiſhed in ſeven years from 1759	5,790
Diminiſhed in 18 years from 1759	33,748
Number of houſes charged in 1750 [*]	690,000
Number of houſes charged in 1710 [†]	729,048

Total

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[*] This reſts on the authority of Dr. Brackenridge. His words are, (in Philoſophical Tranſactions for 1756, page 887) "I have been certainly informed, that from "the ſurvey lately made of the window lights after 1750, "there are about 690,000 houſes charged to that tax in "England and Wales, beſides cottages that pay nothing."

[†] I give this number alſo from the authority of Dr. Brackenridge, who took it himſelf from the accounts of

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Total of houſes according to the Hearth-books of Lady-day, 1690*	1.319,215
Total of houſes from the Hearth-books in 1666†	1.230,000

the Tax Office. See a letter to George Lewis Scot in the Philoſophical Tranſactions, vol. 49, part iſt, p. 270. —By an act of the 8th of *William*, all houſes of 10 windows or more paid 6*s*. over and above 2*s*. for the houſe; and all houſes having 20 windows or more paid 10*s*. In 1710, certain additional duties were laid on all houſes having 10 windows and upwards; and with a view to theſe duties, a new aſſeſſment was ordered, and a return made of the houſes ſubject to an aſſeſſment.—This return could not include in it cottages, nor probably did it include uninhabited houſes.

* This account is given on the authority of Dr. Davenant. See his works, vol. iſt, page 38, where the number of houſes, and alſo of hearths, is given ſeparately for each county.—In page 136 he ſays, that "the "hearth-tax had given a view *certain enough* of the num- "ber of families in the kingdom."

† See Tindall's Continuation of Rapin's Hiſtory, vol. iſt, page 53.—Dr. Davenant ſays, that from 1666 to 1688 there had been about 70,000 new foundations laid. See his works, vol. iſt, page 370.—It is probable that the civil war in the time of King Charles the Firſt, and the emigrations which then took place, leſſened the number of people in the kingdom; and therefore, in in Queen *Elizabeth*'s time, or about the *Reformation*, the number of inhabitants in *England* might have been greater than it was even at the *Revolution*, agreeably to the facts mentioned at the end of my *Appeal to the Public on the Subject of the National Debt*, page 87, &c.

OBSERVATIONS *on the foregoing Accounts.*

Firſt. The firſt of theſe accounts makes the number of houſes in England and Wales in 1777 to be 952,734. Let it, however, be ſtated at a MILLION. Five perſons to a houſe is too large an allowance, as appears from the accounts in page 281. It follows, therefore, that the number of inhabitants in *England* and *Wales* muſt be ſhort of FIVE MILLIONS.

In the kingdom of SWEDEN the number of inhabitants was 2.446,394, in 1763.— In the kingdom of NAPLES (one of the *Two Sicilies)* it was 4.311,503, in 1777.— In all FRANCE, 25.741,320, in 1772*.

Theſe

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* The account here given of *Sweden* is taken from actual ſurveys of the kingdom in 1757, 1760, and 1763. In the firſt of theſe years the inhabitants, of all ages, were found to be 2.323,195; in the ſecond, 2.367,498; in the third, 2.446,394. See a Memoir by M. Wargentin in the 15th vol. of the Collection Academique, printed at Paris, 1772. The account of the kingdom of *Naples* is alſo given from ſurveys made there every year, and publiſhed in the Court Calendars.—In 1766, the number of inhabitants was 3.771,234; in 1772, 4.040·680; in 1777, 4.311,503.

The Intendants of provinces in *France* were, in 1770, 1771, and 1772, ordered to make returns of the number of deaths, births, and marriages in their reſpective diſtricts. The annual average of deaths for theſe three years was 780,040. See a Treatiſe *On the Legiſlation*

and

Theſe facts ſhew, in a ſtriking light, the ſuperiority which arts, commerce, ſcience, induſtry, and liberty give to a people.

ENGLAND

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and Commerce of Corn, printed at Paris in 1775, and tranſlated into Engliſh, and publiſhed in London in 1776, page 42.—I have been aſſured by the ingenious author, now the Director-general of the finances of *France*, that this account may be depended on as rather below the truth; and it affords a deciſive proof that the number of inhabitants in France cannot be leſs than that ſtated above, or 25.741,320, which is the product of the average of deaths multiplied by 33. That this is the leaſt multiplier which ought to be uſed will appear undeniably from the following facts.——In *Sweden*, the average of deaths for 9 years ending in 1763, was 69,125, or a 35th part and two-fifths nearly of the inhabitants. See M. Wargentin's Memoir juſt referred to.—In the kingdom of *Naples*, the average of deaths for 5 years before 1778, was 115,412, or a 37th and a third of the inhabitants.—Theſe facts (and many others of the ſame kind may be found in the Treatiſe on Reverſionary Payments, page 200) convince me that the average of annual deaths in *France* might have been multiplied by 35 inſtead of 33, and this would have brought out the number of inhabitants 27.301,400.—The ſame concluſion nearly may be drawn from the births in *France*, the average of which for five years ending in 1774, was 928,918. See *Recherches ſur la Population de la France, par M. Moheau*, printed at Paris in 1778, page 147.—In *Sweden*, the average of annual births for 9 years, ending in 1763, was 90,240, or a 27th part and a tenth of the inhabitants.—In the kingdom of *Naples*, the average of annual births for 5 years, ending in 1777, was 166,808, or a 25th part and four-fifths of the inhabitants. The medium is $26\frac{1}{2}$, which

ENGLAND does not conſiſt of many more inhabitants than the kingdom of NAPLES; but in reſpect of dignity, weight and force, the kingdom of NAPLES, compared with it, is *nothing*. Not long ago, this little iſland, with its dependencies, like the ſtate of ATHENS formerly among the *Greeks*, was the arbiter of EUROPE, and more than a match for all the three kingdoms I have mentioned, with SPAIN added to them.

Secondly. The great diſparity between the numbers of people in the higher and the lower ranks of life ſeems to deſerve particular obſervation, as it may be collected from the foregoing accounts. Families living in houſes having *ſeven* windows or leſs, muſt conſiſt of perſons in the loweſt ſtations; and yet the number of theſe houſes was 688,903 in 1777. Add to theſe ſuch of the loweſt people as live in the remaining 263,903 houſes; and it will appear, that the people of property and opulence in the ſtate, compared with the reſt, are indeed a

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multiplied by 928,918, gives 24.616,327.——But it is certain, that a greater multiplier than 26½ ought to be uſed in this caſe, becauſe the births exceed the deaths conſiderably leſs in *France* than in either *Sweden* or *Naples*.—Upon the whole, therefore, I reckon that it appears with ſufficient evidence that the inhabitants of *France* may very moderately be ſtated at the number I have given.

very ſmall body. And yet their number is *now* greater in this country than it ever was; and, very probably, it is much greater in this country than in any other *.——It is proper to add, that this obſervation ſhews us diſtinctly why no taxes in a ſtate can be very productive which do not reach the lower as well as higher ranks of people.

But, thirdly, What requires moſt to be attended to is the certain evidence which the preceding accounts give of the progreſs of depopulation in this kingdom.—The number of houſes in ENGLAND and WALES was at the REVOLUTION 1.319,215. The number of houſes now is not a *million*. Our people, therefore, ſince that æra have decreaſed near a *quarter*.—This appears diſtinctly, as far as Dr. Davenant's account is to be depended on †. The following facts and obſervations will confirm this account, and furniſh us with ſome additional evidence on this ſubject.

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* In ENGLAND, the houſes having more than *ſeven* windows are above a *fourth* of all the houſes. In SCOTLAND, the number of houſes having more than *five* windows, and paying the houſe and window duties, was, in 1777, only 16,206; and conſequently could not be above a *fifteenth* of all the houſes.—Agreeably to this poverty, the people of SCOTLAND, though more than a *fifth* of *Britain*, do not contribute more than a *fiftieth* to the revenue.

† Some may ſuſpect that Dr. *Davenant* has, by miſtake, taken from the Hearth-books the number of *houſes*

Firſt. It appears, that there has been a very great decreaſe, ſince the Revolution, in the produce of a tax called the *hereditary and temporary exciſe.* This exciſe (almoſt the only one that exiſted before the Revolution) conſiſts chiefly of 2*s.* 6*d.* per barrel on all ſtrong beer or ale above 6*s.* the barrel, and 6*d.* on every barrel of ale ſold at 6*s.* or leſs; and alſo a duty of 2*s.* 6*d.* per hogſhead on cyder and perry; a duty on mead, ſtrong waters, and low wines and ſpirits. The groſs annual produce of this tax for three years, ending at 1689, was (as appears from the Exciſe books) 740,147*l.* —Its groſs annual produce for four years, ended in 1768, was 527,991*l.* It has decreaſed, therefore, 212,156*l.* *per annum.* One of the reaſons of this decreaſe has been, that in 1736 the duties on low wines and ſpirits (amounting then to 70,000*l.* *per ann.*) were taken from the Hereditary and Temporary Exciſe, and carried to the Ag-

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houſes in the kingdom, when he ought to have taken the number of *families.* But this is improbable; and if true, will make no great difference, as may be inferred from the accounts in page 282.

gregate fund. Deduct*, therefore, 70,000*l.* from 212,156; and the real decreaſe will be 142,156*l.* And this decreaſe will appear more remarkable, when it is conſidered how much leſs the currency and wealth of the kingdom were before the Revolution than they are now.——It may be ſaid, that more wine is now drank; but this, being confined to the higher claſs of people, makes no great difference.—It may with more reaſon be objected, that the lower people drink now greater quantities of ſpirituous liquors, and therefore leſs ale. With reſpect to this, it ſeems ſufficient to obſerve, that it appears from the Exciſe books that the uſe of ſpirtuous liquors never ſunk the produce of this exciſe more than about 40,000*l.* in a year; and that ſince 1751 it has been ſo much checked by new regulations, additional duties, and other cauſes, that moſt probably it does not prevail much more now than it did at the Revolution.

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[*] This is too great a deduction; for the uſe of ſpirituous liquors was in 1736 ſo much increaſed, that it became neceſſary to reſtrain it by additional duties.—The produce of that part of this Hereditary and Temporary exciſe which conſiſts of the tax upon beer only, was 674,387*l.* in 1688; and 694,476*l.* in 1689. See See Dr. Davenant's works, vol. 1ſt. page 175.

After allowing, therefore, for the operation of this cauſe *, (and alſo for the increaſed uſe of wine) there will remain a diminution unaccounted for, of at leaſt 100,000 *l. per annum*.

In conformity to this fact, it appears that there has been a proportionable diminution in the quantity of beer brewed for ſale, and in the number of victuallers.——For three years, ended in 1689, the annual average of ſtrong barrels brewed for ſale was 5.055,870. The average of ſmall barrels was 2.582,248. —For three years, ended in 1768, the former † average was 3.925,131; the latter 1.886,760.

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* The following facts will confirm what is here ſaid, and ſhew the progreſs of gin-drinking in the kingdom. —The uſe of ſpirituous liquors prevailed moſt in 1750 and 1751; and the *annual* average of ſpirits drawn from malted corn, cyder, melaſſes, and brewers' waſh in thoſe two years was 11.326,976 gallons.—In 1767 and 1768 it had ſunk to 3.663,568 gallons.—In 1730 and 1731, it was 6.658,788 gallons.—In 1692 and 1693, it was 2,329,487 gallons.

In 1767 and 1768 the annual average of exciſable brandy imported was 1.612,631 gallons. In 1688 and 1689, it was 1.713,974.

† It is natural to ſuſpect that this decreaſed conſumption of beer muſt have been owing to the increaſe of the taxes upon it. But this does not appear; for in 1761, (after an addition in 1760 of 3*d.* per buſhel to the duty on malt) an addition was made to this tax of 3*s.* per

1.886,760.—The average of common victuallers in the whole kingdom for the former three years * was 47,343; for the latter three years, 34,867.——This laſt fact ſeems of particular conſequence, becauſe victuallers in both periods include all that keep houſes for ſelling any ſtrong liquors; and becauſe alſo there is reaſon to believe, that the private brewery †, of which no account

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per barrel, and yet it produced in the following years rather more in proportion than it did before.—The quantity likewiſe of ſtrong beer brewed for ſale increaſed a little afterwards; though theſe two additions were ſo conſiderable as to bring into the revenue near 900,000*l.* *per annum.*

* For 10 years before the check given to the uſe of ſpirituous liquors in 1751, the victuallers in the kingdom amounted to near 48,000, though the quantity of ſtrong beer brewed annually for ſale was then leſs than it has been for the laſt 15 years. This, I ſuppoſe, muſt have been owing to the vaſt numbers of ſhops for ſelling gin, which, during that period, were opened every where.

† The number of common brewers in the whole kingdom in 1687 and 1689 was 776; in 1767 and 1768 it was increaſed to 1083. One reaſon of this muſt be, that fewer victuallers and private people now brew their own beer.—It is remarkable, that the number of brewers in London *decreaſed* during the ſame period from 187 to 157; and alſo that the quantity of ſmall and ſtrong beer brewed for ſale *decreaſed* from 1.958,859 to 1.533,242 gallons. And this ſeems to confirm what has been already ſuggeſted, that even London is leſs populous now than it was at the Revolution. See page 281.

This

account is taken, was greater formerly than it is now.—I cannot help adding, as a farther fact, indicating a particular degree of populouſneſs at the Revolution, that King William wanting, in 1689, to raiſe 23 new regiments for the war in Ireland, the levies were completed in ſix weeks. See Sir John Dalrymple's Memoirs of Great Britain, vol. 1ſt, page 384.—But what is moſt of all deciſive in the preſent queſtion is, the depopulation which has certainly taken place *lately* in this kingdom.

From the preceding accounts it appears, that between the years 1765 and 1777 a deſtruction has taken place of at leaſt 55,771 houſes having leſs than 8 windows; which is equal to the loſs of above a *quarter of a million* of thoſe inhabitants who furniſh recruits for our navy and army, and trading ſhips; and who, therefore, conſtitute the main ſtrength of the kingdom.

I am not ſenſible that any thing can be objected to the evidence from which this

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This decreaſe was gradual and ſlow till 1726. After 1726 it became conſiderable; and for ſome years before 1750, the quantity of beer conſumed in London was about 100,000 gallons *per annum* leſs than it is now, in conſequence, undoubtedly, of the exceſſive uſe of ſpirituous liquors which then took place in *London* more than any where elſe.

concluſion has been drawn, except that there is an uncertainty in the returns of the cottages. But it ſhould be obſerved,

Firſt, That this uncertainty does not at all affect the evidence for the diminution of houſes *charged* having leſs than eight windows, and of which exact accounts are kept.

Secondly, The returns of the cottages have not, I ſuppoſe, been made with leſs care for 1777 than for 1765; and it is the difference only on which the concluſion I have drawn depends.

But, thirdly, The diminution which there has certainly been in the houſes *charged* having leſs than eight windows, proves undeniably, that there muſt have been a proportionable decreaſe in the cottages *not* charged.

Between the years 1759 and 1765 there appears in the returns, as I have been able to give them, only a diminution of 234 in the houſes charged, and of 6,280 in the cottages. See pages 286, 288. It ſhould not, however, be forgotten, that the increaſe of houſes having *more* than ſeven windows, which took place between 1765 and 1777, muſt have begun before 1765; and that this increaſe ought to be added to the whole decreaſe between 1759 and 1765, in order to determine the whole decreaſe in that period of the meaneſt ſort of houſes; and ſuch an addition would ſhew diſtinctly the rate and progreſs of depopulation among us between the years 1759 and 1777.

Before 1759, no more appears than that the houſes *charged* had diminiſhed 10,851 between 1750 and 1759; and 49,899 between 1710 and 1759. If to this decreaſe is added the decreaſe that muſt have been going on faſter all this time in the houſes *not* charged, it will appear too probable that the depopulation ſince the REVOLUTION cannot have been leſs than I have ſtated it in page 293. It would have been conſiderably greater, had it proceeded from that period conſtantly at the rate with which we know it has proceeded ſince 1765. But there is reaſon to believe that of late it has been much accelerated by particular cauſes, ſome of which will be mentioned preſently.

The Honourable Mr. Grenville, in a pamphlet entitled *Conſiderations on the Trade and Finances of the Kingdom*, after giving the ſame account with that here given of the houſes in *England* and *Wales* in 1759 and 1765, expreſſes the utmoſt ſurprize at the proofs of depopulation which it afforded, and obſerves, " that the deſtruction of " 5790 houſes in ſo ſhort a ſpace as eight " years, is ſuch a ſymptom of diſtreſs as " requires every attention to check the pro- " greſs of the evil.—Relief to the landed " intereſt is now (he adds) no longer the " concern of individuals only who are to " receive that relief, but is become an im- " portant national concern."—What would he have ſaid, had he known that the depopulation which ſhocked him was proceeding ſo rapidly as I have ſhewn; that no attention would be given to it; that the public burdens, inſtead of being leſſened, would increaſe; and that he himſelf had laid the foundation of ſuch an increaſe of them as would, in a few years, bring the nation to the brink of ruin?

The increaſe in the higher claſſes of houſes has been for ſome time obvious to every one. It may be imagined, that this implies ſuch an increaſe of people in the middle and higher ranks of life, as makes amends for the depopulation among the loweſt ranks. But the truth is, that no ſuch concluſion can be drawn. One of the principal cauſes of this increaſe has been that very evil which has deſtroyed the common people, or the increaſe of luxury. This, I think, has been demonſtrated, by the account I have given of London *. See page

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* At Birmingham, in 1700, the inhabitants were 15,042, the houſes 2,504; or 6 to a houſe.—In 1750, the inhabitants were 23,688; the houſes, 4170, or $5\frac{7}{10}$ to a houſe.—In 1770, the inhabitants were 30,804; the houſes, 6,025, or $5\frac{1}{9}$ to a houſe.—In this account we ſee the gradual progreſs of luxury at *Birmingham*; the houſes having increaſed ſo much faſter than the inhabitants, that 600 houſes in 1770 contained no more people than 511 contained in 1700.—See Treatiſe on Reverſionary Payments, page 427.

The following circumſtance may perhaps deſerve ſome notice here.—By the new regulations of the window-tax in 1776 particular inducements were given to divide buildings deemed *ſingle* houſes, but holding *ſeveral* families, into houſes having only one family in each; and this, as well as luxury, may have contributed to increaſe the number of houſes without increaſing the number of inhabitants.

For inſtance. By dividing a houſe having 30 windows, and containing *three* families, into *three* houſes or tenements, having ten windows, and one family in each houſe, only 9*s*. *per annum* would have been ſaved before 1766; but ſince the alteration in the tax that year, 1*l*. 14*s*. *per annum* may be got by ſuch a diviſion.—In like manner. By dividing ſuch a houſe into two houſes, having one family in each, and 15 windows, 3*s*. *per annum* would have been *loſt* before 1766; but now 15*s*. *per annum* may be *ſaved* by it.

N. B.—Before 1766, houſes having from eight to eleven windows paid 1*s*. per window; and houſes having more than eleven windows paid 1*s*. 6*d*. per window, beſides page 280. It muſt, however, be acknowledged, that in many of our towns, and particularly our manufacturing towns, there has been a great increaſe of people as well as of houſes; but it ſhould be conſidered, that it has been derived from the depopulation of country pariſhes and villages, the inhabitants of which, by removing to theſe towns, and many of them thriving there, and living in better houſes, have increaſed the number of ſuch houſes at the expence of meaner houſes. This increaſe of people, therefore, in our towns has either quickened depopulation; or, if not, it muſt have been owing entirely to the increaſe of trade. From the accounts of the exports at the Cuſtom-houſe it appears, that * for ſome years before 1765 they were at the higheſt, and that they

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beſides 3*s.* for the houſe.—By the new regulations in 1766, beſides the old duty of 3*s.* for every houſe, all houſes having ſeven windows pay 2*d.* per window. Houſes having 8, 9, &c. to 13 windows, pay reſpectively 6*d.*—8*d.*—10*d.* &c. to 1*s.* 4*d.* per window.—Houſes having from 14 to 19 windows pay 1*s.* 6*d.* per window. —Houſes having 20, 21, &c. to 24 windows, pay 1*s.* 7*d.* —1*s.* 8*d.* &c. to 1*s.* 11*d.*—Houſes having above 24 windows, pay 2*s.* per window.

* See *The Additional Obſervations on Civil Liberty*, page 113. The annual average of exports for four years ending in 1764, was 15.793,158*l.*—In 1773, the average for nine years had ſunk to 14.814,074*l.* But the imports had

they have ſince decreaſed. This decreaſe, however, has been more than compenſated by the encreaſe of our *home-conſumption*, occaſioned by a vaſt encreaſe* of luxury; and this, though it has operated fatally among the body of the lower people, has, in one way contributed to retard the progreſs of depopulation; I mean, by furniſhing an increaſe of employment, and conſequently of the means of ſubſiſtence, for our manufacturers and artizans. But though depopulation has been thus checked, yet it has proceeded rapidly; and if we aſcribe one half of the increaſe in the higher claſſes of houſes to this cauſe (or a real increaſe of people)

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had increaſed from 10.110,870*l.* to 11.996769*l.*—The decay of foreign trade may farther be underſtood from hence. In 1764, the drawbacks on exportation amounted to 2.264,820*l.*—The average for ten years after 1764 was 1.843,404*l.*—but in 1776 they ſunk to 1.544,300*l.*—In 1777, to 932,860*l.*—In 1778, to 868,600*l.*

* The following account will ſhew how great this increaſe has been.—The *net annual* amount of all the exciſe duties for two years, ending 1768, was 4.431,075*l.* For two years, ending in 1773, it was 4.712,265*l.*—For two years, ending in 1777, it was above FIVE MILLIONS, after deducting the new taxes for 1776 and 1777.—The great increaſe of our importations, while the exportations have decreaſed, as mentioned in the laſt note, is another certain proof of the increaſe of luxury; and has probably been the means of turning the balance of trade againſt us. See *Additional Obſervations on Civil Liberty*, p. 116, &c.

and the other half to luxury, as before explained, we ſhall, I think, reckon very moderately; and it will appear, that in eleven years near 200,000 of our common people have been loſt.

I will only obſerve farther, that ſince the Revolution, moſt of the cauſes of depopulation have prevailed ſo much as to render it an evil which could not but happen. The cauſes I mean are—the increaſe of our navy and army, and the conſtant ſupply of men neceſſary to keep them up—a devouring capital, too large for the body that ſupports it *—the three long and deſtructive continental wars in which we have been involved—the migrations to our ſettlements abroad, and particularly to the Eaſt and Weſt Indies—the engroſſing of farms—the high price of proviſions—but above all, the increaſe of luxury, and of our public taxes and debts.

I have given a particular account of theſe cauſes of depopulation in the Supplement to the Obſervations on Reverſionary Payments,

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* PARIS cannot contain ſo much as a *fiftieth* part of the inhabitants of *France.* LONDON contains a *ninth* of the inhabitants of *England*; and conſumes *annually* about 7,000 perſons, who remove into it from the country every year, but without increaſing it.

page 371, third edition.—I will here only obſerve, that the depopulation they have produced is the more mortifying, becauſe it ſeems, in ſome degree, peculiar to this nation.—In FRANCE, (in the principality of *Dombes*, the dioceſe of *Vaiſon*, and the ſeven generalities of *Auvergne* *, *Lyon*, *Rouen*, *Bourgogne*, *Provence*, and *Alençon*, containing 2152 pariſhes) the average of annual births before 1764 had increaſed in 60 years from 54,827 to 59,894, or in the proportion of 100 to 109.—The average for five years of annual births in the whole kingdom of *France*, (as mentioned in the note, page 281) had been 928,918, in 1774, of which 479,649 were males, and 449,269 females.—The average of deaths, as mentioned in the ſame note, had been 780,040 for *three* years, ending in 1772. But Mr. MOHEAU has given the average for *five* years, ending in 1774 †; and it was 793,931. The annual exceſs of the births above the deaths was, therefore, 134,987; or near a *ſeventh* of the births; and this is probably an exceſs which in *France* more than counterbalances the deſtruction occaſioned by emigration, war, and the ſea-ſervice.

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* See *Recherches ſur la Population*, printed at Paris in 1766, page 274, and page 19, &c. See alſo on this ſubject M. MOHEAU's *Recherches & Conſiderations ſur la Population de la France*, printed at Paris in 1778; where, in page 276, &c. the account of the increaſe of the generalities of Auvergne, Lyon, and Rouen is continued to 1774.

† MOHEAU's *Recherches*, &c. page 65.—The average of marriages was 192,180.

The increaſe in SWEDEN and the kingdom of NAPLES has been diſtinctly mentioned in the note juſt referred to.

In the Engliſh colonies in NORTH AMERICA there has for many years been an increaſe ſcarcely ever before known among mankind.

Thus unhappily diſtinguiſhed are we in this country. Nor will it appear wonderful, when we conſider how unhappily we are diſtinguiſhed by ſome of the worſt cauſes of depopulation; and with what particular force they have been operating for the laſt *twenty* years. At preſent we are ſinking under new incumbrances and difficulties. The moſt valuable of our dependencies are loſt. Another foreign war is begun. Trade is declining; our ſtrength is waſting; and at the ſame time, that load of debts which has preſſed ſo heavily on our population, is increaſing faſter than ever.—Never, certainly, were the reſources of a ſtate ſo anticipated and mortgaged*.—Never before did imprudence and extravagance bring a great kingdom into ſuch peril.

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* The terms of the loan for the preſent year will throw ſome light on what is here ſaid.—A 3 *per cent.* ſtock has been ſold at 40 *per cent.* diſcount, to which has been annexed an Annuity of $3\frac{3}{4}$ *per cent.* for 29 years, reckoned at $11\frac{1}{4}$ years purchaſe, but really worth (when the 3 *per cents.* are at 40 *per cent.* diſcount) $15\frac{14}{100}$ years purchaſe.—The public, therefore, beſides ſubjecting itſelf to the neceſſity of paying at redemption 40*l.* more than it has received for every 100*l.* ſtock, has given a preſent premium on the ſhort Annuity of near 27 *per cent.* And even on theſe terms, (with the profits of a lottery added) only *ſeven millions* could be got, though near *nine millions and a half* were wanted for defraying the neceſſary expences of the year, excluſive of the uſual vote of credit for a million.—Theſe deficiencies muſt be made good; and at leaſt *ten* or *eleven millions* more borrowed within the next eight or nine months, for which, very probably, if the war continues and ſpreads, a higher intereſt and ſtill higher premiums muſt be given.—The national debt is now conſiderably greater than it was in 1776, when Mr. *Hume* wrote the words quoted in the next page; and it is advancing faſt towards *two hundred millions.* It may ſignify little how a nation, in ſuch circumſtances, borrows money; but I am miſtaken if I have not (in the *Supplement* to the Additional Obſervations on Civil Liberty) propoſed regulations by which the loan of this year might have been procured at an intereſt of 5 (or, at moſt, $5\frac{1}{2}$) *per cent.* and conſequently an expence of 100,000*l.* *per annum* for 29 years ſaved; which ſaving, properly applied, might have diſcharged, in 28 years, either the capital of *five millions* bearing *four* per cent. intereſt created in 1777, or a *larger* capital in the *three* per cents.

“ Our late deluſions (ſays Mr. HUME *) “ have much exceeded any thing known in “ hiſtory, not even excepting thoſe of the “ Cruſades. For there is no arithmetical “ demonſtration that the road to the Holy “ Land is not the road to Paradiſe; as there “ is, that the endleſs increaſe of national “ debt is the direct road to national ruin. “ ——So egregious, indeed, has been our “ folly, that we have even loſt all title to “ compaſſion under the numberleſs calami- “ ties that are waiting us.”

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* See Hiſtory of England, vol. 5th, page 475.

May 20, 1779.

POSTSCRIPT.

The Favour of a Friend has lately procured for me, from the Tax-Office, the following Particulars in the Returns for 1759 *and* 1756, *mentioned in Page* 288.

	In 1756.		In 1759.
Houſes *charged* having *leſs* than 10 windows	482,533	—	475,147
Houſes *charged* having from 10 to 14 windows — — —	105,153	—	103,610
Houſes *charged* having from 15 to 19 windows — — —	55,457	—	53,193
Houſes *charged* having 20 windows or *more*	47,559	—	47,199
Total — — —	690,702	—	679,149

This account ſcarcely needs a comment. A compariſon of it with the *returns* in page 285, for 1765 and 1777, will ſhew diſtinctly, that *before* 1759, houſes of all ſorts were decreaſing; but that *before* 1765, an increaſe (produced by increaſed trade and luxury, as explained page 301, &c.) begun among the *higher* claſſes of houſes;

which, *after* 1765, became very conſiderable; but was all along accompanied with a decreaſe much more conſiderable in thoſe inferior claſſes of houſes which conſtitute near *four fifths* of all the houſes in the kingdom.

N. B. In page 288, the year 1750 ſhould have been 1756. The number alſo 690,000 ſhould have been 690,702. And in conformity to theſe corrections, the words in page 300, *diminiſhed* 10,851 *between* 1750 *and* 1759, ſhould be read, *diminiſhed* 11,553 *between* 1756 and 1759.

THE END.

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